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STUDIES IN NONLINEAR OSCILLATION

by

© ERIC D.J. BUCKLEY

A THESIS

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "STUDIES IN NONLINEAR OSCILLATION", submitted by ERIC D.J. BUCKLEY in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

This thesis is devoted to a study of two related problems concerning the second order nonlinear ordinary differential equation

$$x'' + q(t)f(x) = 0 \tag{1}$$

with $q(t) > 0$ and continuous on $[0,\infty)$ and $f(x)$ continuous for all real x and satisfying $xf(x) > 0$ for $x \neq 0$.

The first of these problems is to determine necessary and/or sufficient conditions that the equation (1) admit both oscillatory and nonoscillatory solutions. For such an equation, which we call a "mixed" equation, our second and related problem is to obtain information about the nature of those initial conditions $x(0)$, $x'(0)$ for which the corresponding solution is oscillatory (respectively nonoscillatory) and, in particular, to obtain bounds on the set of those initial conditions for which the corresponding solution never vanishes on $[0,\infty)$.

Our main results regarding the first problem include an extension to equation (1) of a theorem of Jasny and Kurzweil giving sufficient conditions for the existence of an oscillatory solution. Using this extension and known sufficient conditions for the existence of nonoscillatory solutions to equation (1), we obtain both necessary conditions and sufficient conditions on $q(t)$ for equation (1) to be mixed.

With regard to the second problem, we show that under certain conditions on $f(x)$ a solution to (1) satisfying the initial conditions

$$x(0) = a \ , \quad x'(0) = b \qquad (2)$$

can be forced to vanish arbitrarily often on a closed interval $[0,T]$ by choosing $a^2 + b^2$ sufficiently large. It is also shown, under certain conditions on $f(x)$, that the set of values of a for which a solution of (1)-(2) does not vanish on $[0,\infty)$ is of the form $S = [-\alpha,0) \cup (0,\beta]$ where $\alpha,\beta \geq 0$, and that for each $a \in S$, the set of initial slopes b for which the corresponding solution never vanishes is a subset of a closed interval not containing zero. Finally, for a well known special case of equation (1), $f(y) = y^{2n+1}$, we obtain certain inequalities involving the coefficient function $q(t)$ and the initial conditions a,b corresponding to non-vanishing solutions.

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I dedicate this thesis to my wife and children, with a special thanks for their understanding and lack of complaint for the many times when it was necessary for them to make certain sacrifices to the continuation of my studies.

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CHAPTER I

§1.1 Introduction

During the past fifty years the theory of nonlinear oscillations has acquired special importance and has drawn increased interest from many researchers in mathematics, mechanics and physics. Among the family of differential equations forming the subject of this interest, perhaps the most widely studied has been the second order ordinary differential equation

$$x'' + f(t,x,x') = 0 \tag{1.1}$$

in the real domain.

The question of oscillation of extendable solutions to equation (1.1) has been of primary concern to some researchers. By this is meant the resolution of the following problems for equation (1.1): singling out those equations which have at least one oscillatory (respectively nonoscillatory) solution; singling out those equations which have all solutions oscillatory (respectively nonoscillatory); establishing conditions for the existence of bounded (unbounded) oscillatory (non-oscillatory) solutions; establishing conditions for the existence of solutions with decreasing amplitudes; estimating the amplitude, the number of zeros on a given interval, the distance between consecutive zeros, and so on, of a solution to equation (1.1). Here, an extendable solution is called oscillatory if it exists on the interval $[o,\infty)$ and has infinitely many zeros accumulating at ∞, otherwise it is called nonoscillatory.

More recently there has developed an interest in obtaining methods for solving the following problem for that class of equations for which not all solutions can be oscillatory (nonoscillatory): to determine, for a given equilibrium state x_r, the set $\{(x(t_o),x'(t_o))\}$ of initial values corresponding to oscillatory solutions of the equation with respect to x_r, that is, solutions for which the function $x(t)-x_r$ has an infinite number of zeros accumulating at ∞.

The results so far obtained from research into the above questions have not as yet provided complete answers to all of them, but nevertheless have practical significance in that they are concerned with the properties of equations that are often encountered in various areas of natural science and applications. We shall cite as examples certain equations, fully referenced by V.N. Shevyelo [20] for which a knowledge of their oscillatory properties is of practical interest.

The equation

$$\frac{d}{dr}\left(r^{\rho}\frac{du}{dr}\right) \pm r^{\lambda}f(u) = 0 \tag{1.2}$$

is encountered in astrophysics in the form of Emden's equation, in atomic physics in the form of the Fermi-Thomas equation, and in fluid mechanics in the form of the equation

$$\frac{d}{dr}\left(r^{2}\frac{du}{dr}\right) + r^{\lambda}f(u) = 0\ , \quad \lambda > 0\ , \quad r > 0\ . \tag{1.3}$$

A series of works studies the properties of particular cases of equation (1.2). The results of the investigations of R. Fowler, published during the period 1915-1931, are summarized in the monograph of

R. Bellman [2], (Chapter 7). M. Chimino [6] showed that in one special case of equation (1.3), the zeros of the solution, $u(r)$, correspond to certain equilibrium states of a fluid.

In relativistic mechanics the following equation, introduced by N.S. Kalitsin (c.f. Shevyelo [20]) is encountered:

$$\frac{d}{dt}\left[\frac{m(t)\ \vec{v}}{(1-v^2/c^2)^{1/2}}\right] - \frac{\vec{u}}{(1-u^2/c^2)^{1/2}}\ \frac{dm(t)}{dt} = \vec{F}\ . \qquad (1.4)$$

The equation (1.4) can describe the movement of certain radioactive particles having a velocity close to the velocity of light. The work of V.N. Shevyelo and V.G. Shtelik [21] is devoted to a study of the oscillatory properties of solutions to certain particular cases of equation (1.4).

Equations of Lane-Emden type of the form

$$\frac{d^2x}{dt^2} + \frac{2}{t}\,\frac{dx}{dt} + f(x) = 0 \qquad (1.5)$$

were probably first studied by Emden [8] in examining the thermal behaviour of spherical clouds of gases acting in gravitational equilibrium and subject to the laws of thermodynamics. A more recent treatment of the case with $f(x) = x^n$ is available in Chandrasekhar [3]

The phase motion of charged particles in a synchrocyclotron is described by the equation

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\ \frac{d\phi}{dt}\right) + \frac{eV}{2\pi}\sin\phi = \frac{eV}{2\pi}\sin\phi_s\ , \qquad (1.6)$$

where E_s, ω_s, K, e and V are certain given parameters.

Physicists have called attention to the fact that equation (1.6) coincides with the equation of motion of a pendulum of variable length and mass moving under the influence of the force of gravity and a constant rotational moment of magnitude such that the position of stable equilibrium of the pendulum is changed from $\phi = 0$ to $\phi = \phi_s$. The equation (1.6) is a particular case of the equation

$$\frac{d}{dt}\,[m(t)\dot{\psi}] + f(t)u'(\psi) = 0 \;, \tag{1.7}$$

where

$$u'(\psi) = \frac{du}{d\psi} = \frac{-1}{\sin \phi_s}\,[\cos\,(\phi_s+\psi) - \cos\,\phi_s] \;.$$

Equation (1.7) describes the phase motion of charged particles in the electromagnetic field of those accelerators whose operation satisfies the principle of phase stability, that is, accelerators which are designed to pass the charged-particle beam through each successive accelerating gap at the same equilibrium phase of the accelerating field. In the theory of accelerators there arises the important question of determining the "region of capture" - the set of initial conditions corresponding to solutions of equation (1.7) that are oscillatory with respect to the equilibrium position $\phi = \phi_s$. Knowledge of the region of capture is necessary for the determination of the fraction of injected particles which are trapped in the synchronous regime of the accelerator and consequently reach an energy very close to the theoretically calculated energy.

For a more detailed account of applications of second order

nonlinear oscillation one may refer to the survey article by Shevyelo [20] where a complete bibliography up to 1963 is given.

A particular case of equation (1.1), for which the question of oscillation of solutions has been thoroughly studied, is the equation

$$x'' + f(t,x) = 0 \tag{1.8}$$

where $f(t,x)$ is a continuous function of the variables $t \geq 0$ and $|x| < \infty$. Theorems on oscillation and nonoscillation of solutions to (1.8), together with a comprehensive survey of current literature in this area, may be found in the survey by Wong [25] in which the Shevyelo bibliography is updated to the year 1967. The paper of Wong, and indeed a considerable portion of recent research articles dealing with equations of the type (1.8), was inspired by a well known paper of Atkinson [1] which deals with the special case of equation (1.8),

$$x'' + q(t)\, x^{2n+1} = 0 \ , \quad n = 1,2,3,\ldots \tag{1.9}$$

Although Atkinson's work was later extended, e.g. by Waltman [23], Macki and Wong [15], Das [7], Nehari [17], Moore and Nehari [16] and others, there can be little doubt that it provided the stimulus for subsequent investigations by numerous authors into the qualitative theory of second order nonlinear oscillation, which includes the resolution of those problems referred to in the second paragraph of this section.

§1.2 Statement of the problem

It is not the purpose of this presentation to attempt any further

extension of results relating to the broad question of oscillation of solutions to equation (1.1). We shall be concerned rather with the following two related problems as they pertain to a specific case of equation (1.1), namely the equation

$$x'' + q(t)f(x) = 0 , \qquad (1.10)$$

where $q(t)$ is non-negative and continuous for $t_0 \leq t < \infty$, while $f(x)$ is "strictly nonlinear", continuous for $|x| < \infty$, and satisfies

$$xf(x) > 0 \quad \text{for} \quad x \neq 0 .$$

These two problems are:

A. to determine necessary and/or sufficient conditions for equation (1.10) to admit both oscillatory and nonoscillatory solutions; and

B. given an equation of the type (1.10) which admits both oscillatory and nonoscillatory solutions, to determine the set of initial conditions $\{(x(t_0),x'(t_0))\}$ corresponding to oscillatory (nonoscillatory) solutions.

Problems A and B have been previously considered by Moore and Nehari [16], Nehari [17] and Hinton [11]. Moore and Nehari have demonstrated examples of equation (1.10) having both oscillatory and nonoscillatory (in fact, zero-free) solutions, while Hinton has exhibited a bound $L = L(t_0,q(t))$ such that any solution to (1.9) satisfying

$$|x(t_0)| > L$$

is oscillatory on $[t_0,\infty)$, providing that $q(t)$ satisfies certain differentiability and integrability conditions.

It would appear reasonable that any attempt to obtain a complete solution to problem B should proceed as an extension to the solution of problem A, and that in the absence of conditions which are both necessary and sufficient for equation (1.10) to admit oscillatory and nonoscillatory solutions, one can expect to provide, at best, a limited solution to problem B. In the following chapters, therefore, we shall be concerned with both problems, sometimes individually, at other times jointly. Chapter II will include some new results and a summary of known results pertaining to problem A, as well as a complete discussion of problem B as it relates to a particular equation

$$x'' + [4(t+1)^{n+2}]^{-1}\, x^{2n+1} = 0 \ . \qquad (1.11)$$

Chapter III will deal almost exclusively with problem A and contains new results which place further restrictions on the class of functions $q(t)$ for which equation (1.9) or (1.10) can admit both oscillatory and nonoscillatory solutions. In Chapter IV we shall show, motivated by example (1.11), that the set of initial conditions generating certain types of solutions will, for the general equation (1.10), satisfy the same properties as it does in the case of the example (1.11). We do not, however, succeed in characterizing the set of initial conditions generating oscillatory solutions to (1.10).

CHAPTER II

§2.1 Existence, uniqueness and extendability of solutions.

Consider the second order ordinary differential equation

$$x'' + q(t)x^{2n+1} = 0 \ , \quad n \text{ a positive integer,} \tag{2.1}$$

and its generalization

$$x'' + q(t)f(x) = 0 \ , \tag{2.2}$$

where it is assumed that

$$q(t) \text{ is positive and continuous on } [0,\infty) \ , \tag{2.3}$$

$$f(x) \text{ is continuous for } x \text{ in } (-\infty,\infty) \text{ and } xf(x) > 0 \text{ for } x \neq 0 \ , \tag{2.4}$$

and

solutions to initial value problems for (2.2) are unique. (2.5)

(In what follows, additional conditions will occasionally be placed on $f(x)$ in order to generalize to equation (2.2) certain results which hold for (2.1).)

By assumption (2.5), for any real numbers a and b, the equation (2.2) has, on some interval I, $0 \in I \subset [0,\infty)$, a unique solution $x = x(t;a,b)$ satisfying

$$x(0) = a \ , \quad x'(0) = b \ , \tag{2.6}$$

and this solution can be extended uniquely to a maximal interval of

existence $J \subset [0,\infty)$.

One is immediately concerned with determining conditions which will guarantee that $J = [0,\infty)$. The fact that this is not always the case has been demonstrated by Coffman and Ullrich [5] with an example of the equation

$$x'' + q(t)x^3 = 0 \tag{2.7}$$

in which $q(t)$ is locally of bounded variation everywhere on $[0,\infty)$ with the exception of one point t_o , and such that at least one solution of (2.7) has $[0,t_o)$ as its maximal interval of existence in $[0,\infty)$. This example shows that global existence of solutions of (2.1) can be destroyed by a pathology of the coefficient at a single point, and in fact even when the coefficient differs from a constant by an arbitrarily small amount.

It is observed by Coffman and Ullrich that a solution of (2.1) on an interval $[0,t_o)$ can fail to have a continuation to the right of t_o only if the solution changes sign infinitely often as t approaches t_o from the left. The same remains true for equation (2.2). Indeed, since a solution $x(t)$ of (2.2) always satisfies $xx'' \leq 0$, an elementary argument shows that for a solution $x(t)$ defined on the interval $[0,t_o)$ and having only finitely many zeros there, both x and x' possess finite limits as $t \to t_o^-$.

The following theorem, which is a generalization of a theorem of Coffman and Ullrich, gives conditions sufficient to guarantee that all solutions of (2.2)-(2.6) exist on $[0,\infty)$.

Theorem 2.1

Let (2.3), (2.4) and (2.5) be satisfied, and assume, in addition, that

$$q(t) \text{ is locally of bounded variation on } [0,\infty) . \tag{2.8}$$

Then for any real numbers a and b, the initial value problem (2.2) - (2.6) has a unique solution which exists on $[0,\infty)$.

Proof.

Let $t_1 > 0$ and choose $m > 0$ such that $q(t) \geq m > 0$ on $[0,t_1]$. It is possible to approximate $q(t)$ uniformly on $[0,t_1]$ by a sequence of functions $\{p_k(t)\}$ where each $p_k(t)$ is of class C' on $[0,t_1]$ and satisifes*

$$p_k(t) \geq m \quad \text{on} \quad [0,t_1] \tag{2.9}$$

and

$$\int_0^{t_1} |p_k'(t)|\,dt \leq T \tag{2.10}$$

where T is the total variation of $q(t)$ on $[0,t_1]$. For each $k = 1,2,3,\ldots$ let $x_k(t)$ be the solution of

$$x'' + p_k(t)f(x) = 0$$

which satisfies

$$x_k(0) = a , \quad x_k'(0) = b .$$

Suppose $x_k(t)$ exists on an interval $[0,t_2)$ where $0 < t_2 < t_1$.

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* See Appendix

Define

$$\phi_k(t) = (x_k'(t))^2 + 2p_k(t)\int_0^{x_k(t)} f(u)du \ , \tag{2.11}$$

for $0 \leq t < t_2$. Then $\phi_k(t)$ is of class C' on $[0,t_2)$ and it follows from (2.9) and (2.4) that $\phi_k(t) \geq 0$. Moreover

$$\phi_k'(t) = 2p_k'(t) \int_0^{x_k(t)} f(u)du \ , \quad 0 \leq t < t_2 \ .$$

It follows from (2.9) that

$$\frac{p_k'(t)}{p_k(t)} \leq \frac{|p_k'(t)|}{m} \tag{2.12}$$

and therefore

$$\phi_k'(t) \leq \frac{|p_k'(t)|}{m} [x_k'(t)^2 + 2p_k(t) \int_0^{x_k(t)} f(u)du]$$

or

$$\phi_k'(t) \leq \frac{|p_k'(t)|}{m} \phi_k(t) \ ; \quad 0 \leq t < t_2 \ . \tag{2.13}$$

Integrating the inequality (2.13) yields

$$\phi_k(t) \leq \phi_k(0) \exp \left(\int_0^t \frac{|p_k'(s)|ds}{m} \right) \ ; \quad 0 \leq t < t_2$$

which, in view of (2.12), implies

$$\phi_k(t) \leq \phi_k(0) \exp \left(\frac{T}{m}\right) \ ; \quad 0 \leq t < t_2 \ .$$

It therefore follows from (2.11) that $x_k'(t)$ is bounded on $[0,t_2)$ and

thus $|x_k(t)| \leq \sup_{[0,t_2)} |x_k'(t)| \cdot t$ on $[0,t_2)$. But this implies that the solution $x_k(t)$ can be continued to the right of t_2 and, since $t_2 \in (0,t_1)$ was arbitrary, it follows that the solution $x_k(t)$ exists on $[0,t_1]$ for each finite $t_1 > 0$.

The sequence $x_k(t)$ has a subsequence which converges uniformly on $[0,t_1]$ to a solution $x(t)$ of (2.2)-(2.6) (Hartman [10], page 14, Theorem 3.2). This shows that (2.2)-(2.6) has a solution on $[0,t_1)$ for arbitrary $t_1 > 0$, and this solution is unique by (2.5).

§2.2 Results pertaining to problem A

Assume conditions (2.1) through (2.5) are satisfied. Let F be the set of those functions $q(t)$ satisfying (2.3) for which the conclusion of Theorem 2.1 holds for the equation (2.1), and let F' be the set of those functions $q(t)$ satisfying (2.3) for which the conclusion of Theorem 2.1 holds for the equation (2.2). We shall say that $q \in O \subset F$ (respectively $q \in O' \subset F'$) if all non-trivial solutions of (2.1) (respectively (2.2)) are oscillatory; that $q \in N \subset F$ (respectively $q \in N' \subset F'$) if all non-trivial solutions of (2.1) (respectively (2.2)) are nonoscillatory finally, that $q \in M \subset F$ (respectively $q \in M' \subset F'$) if the equation (2.1) (respectively (2.2)) admits both non-trivial oscillatory solutions and non-trivial nonoscillatory solutions. For a given $q \in F$ (respectively $q \in F'$) we shall say that equation (2.1) (respectively (2.2)) is oscillatory, non-oscillatory, or mixed, according as $q \in O$, $q \in N$, or $q \in M$ (respectively $q \in O'$, $q \in N'$, or $q \in M'$).

It is well known that each of the above defined sets is non-empty. One may refer to the Wong paper [25] for a summary of the known sufficient conditions for membership of a function $q(t)$ in each. The

following important results are reproduced here because of their direct bearing on our later work.

Theorem 2.2 (Atkinson [1])

A necessary and sufficient condition that $q \in \mathcal{O}$ is

$$\int_0^\infty tq(t)dt = +\infty . \tag{2.14}$$

If $q \in C'[0,\infty)$ with $q'(t) \geq 0$ then

$$\int_0^\infty t^{2n+1}q(t)dt < +\infty \tag{2.15}$$

is a sufficient condition that $q \in \mathcal{N}$.

Theorem 2.3 (Wong [25])

Let $f(x)$ be such that for some $p > 1$,

$$\liminf_{|x| \to \infty} \frac{|f(x)|}{|x|^p} > 0 . \tag{2.16}$$

Then (2.14) is a necessary and sufficient condition that $q \in \mathcal{O}'$.

In the paper [25] Wong speculated that Theorem 2.3 remains valid if (2.16) is replaced by the weaker conditions that for some $\alpha \geq 0$,

$$\int_\alpha^\infty \frac{dx}{f(x)} < \infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{dx}{f(x)} < \infty . \tag{2.17}$$

It was shown by Macki [14], however, that in the absence of (2.16) condition

(2.17) is not sufficient for $q \in \mathcal{O}'$. Whether it is necessary is still an open question. In a joint paper of Macki and Wong [15] it is shown that Theorem 2.3 does remain valid under the additional condition that $f(x)$ is nondecreasing (or that $f(x)$ is bracketed between two non-decreasing functions that satisfy (2.17)). Thus

Theorem 2.4 (Macki and Wong [14])

If (2.17) is satisfied and if

$$f \in C'(-\infty,\infty) \quad \text{with} \quad f'(x) \geq 0 \tag{2.18}$$

then (2.14) is necessary and sufficient that $q \in \mathcal{O}'$.

Theorem 2.5 (Nehari [18])

If

$$q(t)\,(t \log t)^{n+2} \text{ is non-increasing} \tag{2.19}$$
$$\text{for } t \geq T \geq 0 ,$$

then $q \in \mathcal{N}$.

Theorem 2.6 (Jasny [12])

If $q(t)$ is absolutely continuous on each finite interval $[t_1,t_2]$, $t_1 \geq a$, and if

$$(n+2)q(t) + tq'(t) \geq 0 \quad \text{a.e. for } t \geq t_o , \tag{2.20}$$

$$\sqrt{t} \int_t^\infty q(t) \left\{ \int_t^\infty du \int_u^\infty q(s)ds \right\}^{-\frac{2n+1}{2n}} dt \leq M \tag{2.21}$$

$$\text{for } t > t_o , \text{ where } M = \text{constant},$$

then $q \notin N$ (i.e., (2.1) admits an oscillatory solution).

Theorem 2.6 of Jasny was later improved by Kurzweil [13] as follows:

Theorem 2.7 (Kurzweil [13])

If

$$t^{n+2}q(t) \text{ is positive and non-decreasing for } t \geq t_o \tag{2.22}$$

then $q \notin N$.

The improvement effected by Kurzweil consisted primarily in showing that unless q belonged to class O , (in which case the theorem holds trivially) condition (2.20) implies (2.21). Furthermore, since (2.20) is required only almost everywhere, the restriction of absolute continuity on q can be dropped when (2.22) is required in place of (2.20). We now show that Theorem 2.6 can be generalized to equation (2.2).

§2.3 Generalization of a theorem of Jasny.

Let (2.3), (2.4) and (2.5) hold, and assume the conclusion of Theorem 2.1. Suppose further that

$$\int_o^{\pm\infty} \frac{dx}{f(x)} = +\infty \tag{2.23}$$

and let the conditions (2.18) and (2.17) for all $\alpha > 0$ be satisfied. Then, according to Theorem 2.4 either $q \in O'$ or $\int_o^\infty tq(t)dt < \infty$,

and in the latter case the function

$$\phi(t) = \int_t^\infty ds \int_s^\infty q(\tau)d\tau \tag{2.24}$$

is well defined and exists on $[0,\infty)$. Moreover (2.17) and (2.23) imply that the function $H(u) = \int_u^\infty \frac{dx}{f(x)}$ is defined and strictly decreasing for $u \in (0,\infty)$, and satisfies

$$\lim_{u \to o^+} H(u) = +\infty , \quad \lim_{u \to \infty} H(u) = 0 .$$

Therefore H has an inverse function h defined on $(0,\infty)$ satisfying

$$\lim_{t \to \infty} h(t) = 0 , \quad \lim_{t \to o^+} h(t) = +\infty$$

and $h(t)$ is strictly decreasing on $(0,\infty)$. Let $tq(t) \in C'[0,\infty)$ and define

$$F(t,u) = 2(tq(t))' \int_o^u f(\tau)d\tau + q(t)uf(u) \quad . \tag{2.25}$$

<u>Theorem 2.8</u>

Under the above assumptions, let

(a) $\int^\infty q(t)f(\alpha t)dt = +\infty$ for any $\alpha > 0$.

(b) $F(t,u) \geq 0$ for $t \geq t_o$, all real u ,

(c) $\sqrt{t} \int_t^\infty q(s)f(h(\phi(s)))ds \leq M = \text{const.}$ for $t \geq t_o$.

Then $q \notin N'$ (i.e., (2.2) has an oscillatory solution).

Before beginning the proof, we remark that Theorem 2.8 is a generalization of Theorem 2.6 for the case of $tq(t) \in C'[0,\infty)$. When applied to equation (2.1), the condition (c) of Theorem 2.8 reduces to (2.21) and condition (b) reduces to (2.20). In the case of equation (2.1) condition (a) follows from (b) (see Jasny [12]).

The proof of the theorem depends on the following lemma.

Lemma. If $x(t)$ is a solution to (2.2) such that $x(t)$ is monotone and of one sign for $t \geq t_o$, and if $\lim_{t \to \infty} x'(t) = 0$ then

$$|x'(t)| < \int_t^\infty q(s)f(h(\phi(s)))ds \ , \quad t > t_o \ .$$

Proof of lemma

It may be assumed that $x(t) > 0$ for $t > t_o$ (for, if not, then an identical argument can be applied to the function $y(t) = -x(t)$ which satisfies the equation $y'' + q(t)g(y) = 0$, where $g(y) = -f(-y)$ and g also satisfies our hypotheses). Let $s \geq t \geq t_o$ and we have from (2.2)

$$x'(t) - x'(s) = \int_t^s q(\tau)f(x(\tau))d\tau$$

from which

$$x'(t) = \int_t^\infty q(\tau)f(x(\tau))d\tau \ . \qquad (2.26)$$

Now x is strictly increasing in t, so it follows from (2.18) that $f(x(\tau)) \geq f(x(t))$ for $\tau > t$. Therefore

$$\frac{x'(t)}{f(x(t))} > \int_t^\infty q(\tau)d\tau \quad .$$

Integrating the last inequality over $[t,\infty)$,

$$\int_t^\infty ds \int_s^\infty q(\tau)d\tau < \int_t^\infty \frac{x'(s)}{f(x(s))} ds = \int_{x(t)}^L \frac{d\tau}{f(\tau)} \leq \int_{x(t)}^\infty \frac{d\tau}{f(\tau)}$$

where $\lim_{t \to \infty} x(t) = L \leq +\infty$. Therefore

$$\int_t^\infty ds \int_s^\infty q(\tau)d\tau < H(x(t)) \ .$$

Since h is strictly decreasing, it follows that

$$h[H(x(t))] < h\left[\int_t^\infty ds \int_s^\infty q(\tau)d\tau\right] = h(\phi(t))$$

or

$$x(t) < h(\phi(t)) \ .$$

Thus, from (2.26) we have

$$|x'(t)| < \int_t^\infty q(\tau)f(h(\phi)\tau)))d\tau \ , \qquad t > t_o$$

which is the conclusion of the lemma.

<u>Proof of Theorem 2.8</u>.

Let the solution $x(t)$ of (2.2) have initial conditions

$$x(t_o) = 0 \ , \quad x'(t_o) = K \ .$$

We will show that if $K > 0$ is sufficiently large then the solution oscillates. Suppose, on the contrary, that $x(t)$ is positive for $t > t_o$. Then $x(t)$ must satisfy $\lim_{t \to \infty} x'(t) = 0$. For if not, then $\lim_{t \to \infty} x'(t) = \beta$ where $0 < \beta < \infty$. Since $x'(t)$ is decreasing, given $\epsilon > 0$ there exists $T > t_o$ such that $\beta \leq x'(t) \leq \beta + \epsilon$ for $t \geq T$, and there exists α, $0 < \alpha < \beta$ such that

$$x(t) \geq x(T) + \beta(t-T) > \alpha t \ , \quad t \geq T \ .$$

But then we have from (2.2)

$$x'(T) = x'(t) + \int_T^t q(s)f(x(s))ds \geq$$

$$\geq x'(t) + \int_T^t q(s)f(\alpha s)ds \ .$$

Letting $t \to \infty$ we obtain a contradiction of hypothesis (a) of the theorem. Thereore $\lim_{t \to \infty} x'(t) = 0$. Now choose K so large that

$$K = x'(t_o) > \frac{M}{\sqrt{t_o}} \ . \tag{2.27}$$

From the lemma and the fact that $x'(t)$ is continuous, there exists a point $t_1 > t_o$ such that $x(t_1) = 0$. Define

$$V(t) = t(x'(t)^2 + 2q(t) \int_o^{x(t)} f(\tau)d\tau) - x(t)x'(t) \ .$$

Then

$$V'(t) = 2(tq(t))' \int_0^{x(t)} f(\tau)d\tau + q(t)x(t)f(x(t))$$

$$V'(t) = F(t,x(t)) \geq 0 \quad \text{for} \quad t \geq t_o \ .$$

Therefore $V(t)$ is non-decreasing, and

$$0 \leq V(t_1) - V(t_o) = t_1 x'(t_1)^2 - t_o x'(t_o)^2 \ .$$

Thus

$$t_1 x'(t_1)^2 \geq t_o x'(t_o)^2$$

and it follows from (2.27) that at the point t_1 we also have

$$|x'(t_1)| \geq \sqrt{\frac{t_o}{t_1}} \, |x'(t_o)| > \frac{M}{\sqrt{t_1}} \ .$$

Again, by the lemma, there exists a point $t_2 > t_1$ such that $x(t_2) = 0$. Continuing in this manner we establish the existence of an infinite sequence $\{t_k\}$ of points such that $x(t_k) = 0$, $k = 0,1,2,\ldots$, contradicting the supposition that $x(t)$ is eventually monotone.

§2.4 An example

As a result of the theorems of §2.2 one can construct infinitely many coefficients $q(t)$ belonging to each of the classes $\mathcal{O}$, $\mathcal{M}$ and $\mathcal{N}$. In fact if one defines the set

$$\mathcal{A} = \{q(t) = (t+1)^{-\gamma}/\gamma \text{ is real}\} \subset \mathcal{F}$$

then

(a) $A \subset O$ for $\gamma \leq 2$

(b) $A \subset M$ for $2 < \gamma \leq n+2$

(c) $A \subset N$ for $n+2 < \gamma$

Statement (a) follows from Theorem 2.2, statement (b) from Theorems 2.2 and 2.7 while statement (c) follows from Theorem 2.5.

One might be tempted, on the basis of Theorem 2.2, to speculate that if both (2.14) and (2.15) fail, then $q(t)$ belongs to class M. This, of course, is not true, for in the case $n+2 < \gamma \leq 2n+2$ both (2.14) and (2.15) fail for the function $q(t) = (t+1)^{-\gamma}$, yet $q(t) \in N$ as asserted in (c) above.

Once it is established that a certain function $q(t)$ falls in class M' it becomes natural to ask the question

> "does the set $\{(x(0),x'(0))\}$ of inital conditions which generate oscillatory (nonoscillatory) solutions possess certain topological properties?"

Moore and Nehari [16] give an example of a function $q(t) \in M$ which serves to illustrate that the set of initial conditions which generate oscillatory solutions is not always a closed set. Their example is deserving of comment at this point because it serves to indicate certain directions for research into the question of the dependence of oscillation (nonoscillation) on initial conditions.

Their example involves the equation

$$x'' + \frac{1}{4}(t+1)^{-n-2}x^{2n+1} = 0 . \tag{2.28}$$

Equation (2.28) has, as its general solution,

$$x(t) = (t+1)^{1/2}\, u[\log\,(t+1)] \tag{2.29}$$

where $u(t)$ is the general solution of

$$4u'' + u^{2n+1} - u = 0\ . \tag{2.30}$$

All solutions of (2.30) are solutions of

$$4(u')^2 + (n+1)^{-1}\, u^{2n+2} - u^2 = A \tag{2.31}$$

where

$$A = 4u'(0)^2 + (n+1)^{-1}\, u(0)^{2n+2} - u(0)^2\ . \tag{2.32}$$

It is shown by Moore and Nehari that:

(i) If $A = 0$ then all solutions of (2.31) are given by $\pm\, u(t+\tau)$ for any τ, where $u = u(t)$ has the asympotic behaviour

$$u(t) = (n+1)^{1/2n}\, \exp[-(t-t_1)(2+\gamma)^{-1}](1+o(1))$$

as $t \to \infty$, and $\gamma > 0$, $t_1 > 0$. All such solutions are of constant sign.

(ii) If $A > 0$, $u(t)$ has an infinite number of zeros in $[0,\infty)$ and is in fact periodic.

(iii) If $A < 0$, $u(t)$ is periodic, and $u(t)$ oscillates infinitely often about the line $u \equiv 1$ or $u \equiv -1$ without intersecting the horizontal axis.

It is clear from (i) and (iii) that if $A \le 0$ then the solution (2.29) of (2.28) is nonoscillatory, and from (ii) that if $A > 0$ then it is oscillatory.

Suppose a solution $x(t)$ of (2.29) is required to satisfy the initial conditions

$$x(0) = a , \quad x'(0) = b . \tag{2.33}$$

Then

$$u(0) = a , \quad u'(0) = b - \frac{1}{2} a$$

and therefore

$$A = 4(b - \frac{a}{2})^2 + \frac{a^{2n+2}}{n+1} - a^2 . \tag{2.34}$$

Thus the oscillation or nonoscillation of $x(t)$ depends on whether $A > 0$ or $A \leq 0$ respectively. We consider each of the three possible cases.

It follows from (2.34) that the case $A < 0$ corresponds to

$$|b - \frac{a}{2}| < \frac{|a|}{2} \left(1 - \frac{a^{2n}}{n+1}\right)^{1/2} , \quad 0 < |a| \leq (n+1)^{1/2} ,$$

while the case $A = 0$ corresponds to

$$|b - \frac{a}{2}| = \frac{|a|}{2} \left(1 - \frac{a^{2n}}{n+1}\right)^{1/2n} , \quad |a| \leq (n+1)^{1/2n} .$$

In the real plane, $R^2 = \{(a,b) / a,b$ are real$\}$, the case $A = 0$ corresponds to the set V_o of points (a,b) which lie on the smooth curve illustrated in figure 1 below, while the case $A < 0$ corresponds to the set $V\_$ of points (a,b) lying in the interior of the region bounded by V_o .

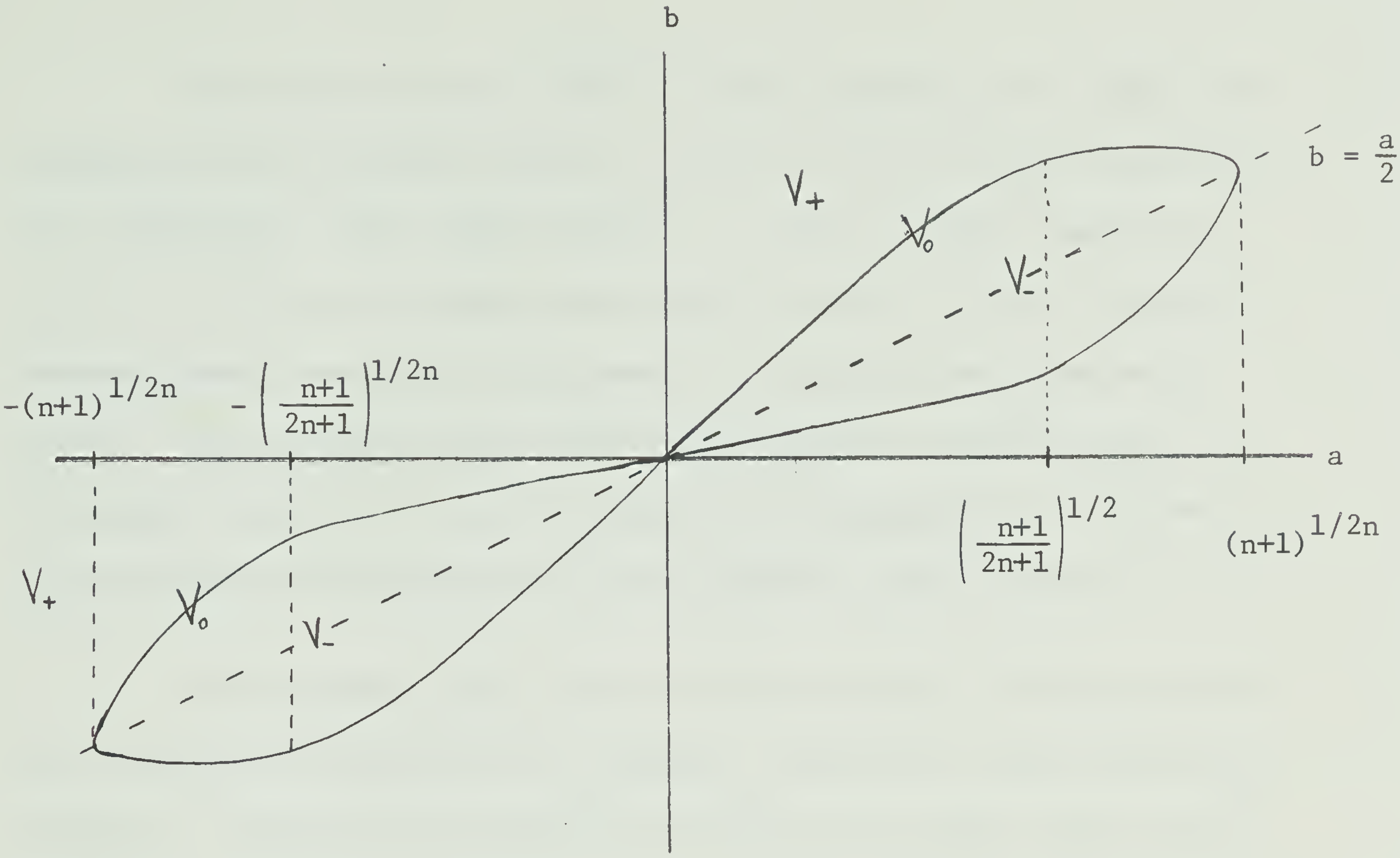

FIGURE I

The curve consisting of the points in V_o is symmetric with respect to the origin. It intersects the line $b = \frac{a}{2}$ at the origin and at $\pm\,(n+1)^{1/2n}$, and assumes its maximum vertical deviation from the line $b = \frac{a}{2}$ at the points $\pm\,(\frac{n+1}{2n+1})^{1/2n}$. Thus it fails to be symmetric with respect to the line $b = \frac{a}{2}$.

The case $A > 0$ corresponds to the inequalities

$$\left|b - \frac{a}{2}\right| < \frac{|a|}{2}\left(1 - \frac{a^{2n}}{n+1}\right)^{1/2}, \quad \text{for } 0 < |a| \le (n+1)^{1/2n} ;$$

$$\text{all } (a,b), \quad \text{for } (n+1)^{1/2n} < |a| .$$

In figure 1 the case $A > 0$ corresponds to the set of points V_+ lying in the exterior of the closed region bounded by V_o.

Thus the solution $x(t)$ of (2.28) satisfying the initial conditions (2.33) is nonoscillatory if $(a,b) \in V_{-} \cup V_{o}$ and is oscillatory if $(a,b) \in V_{+}$. For $(a,b) \in V_{-} \cup V_{o}$, $x(t)$ is always negative if $a < 0$, $b < 0$ and is always positive if $a > 0$, $b > 0$. Thus all nonoscillatory non-trivial solutions of (2.28) are free of zeros in $[0,\infty)$. It is not yet known whether there exists a function $q(t) \in M$ for which equation (2.1) has both zero-free nonoscillatory solutions and nonoscillatory solutions with a finite positive number of zeros.

Our comment, above, that the set of initial conditions generating oscillatory solutions is not always a closed set is made evident by figure 1. The figure also demonstrates that for a given initial value $a = x(0)$, the set of initial slopes $b = x'(0)$ which yield oscillatory solutions is not always a connected set. In fact, for certain values of a, there exists a positive number $\delta = \delta(a)$ with the property that if $b \in (-\infty,\delta)$ then the solution of (2.28)-(2.33) oscillates, while if $b = \delta(a)$ the solution is free of zeros on $[0,\infty)$, whereas if b is sufficiently large the solution again oscillates. We will show in a later chapter that under rather general conditions of $f(x)$, if equation (2.2) admits any zero-free solutions, then the set of initial values which generate these solutions is bounded in a manner somewhat similar to that in the above example.

CHAPTER III

§3.1 Properties of a mixed equation

Any consideration of the manner in which oscillation or non-oscillation of solutions is determined by the initial conditions must proceed on the assumption that the equation under consideration admits solutions of both types. It is therefore adviseable at this point to consider further the question of the existence of both types of solutions.

For the equation

$$x'' + q(t)x^{2n+1} = 0 \tag{2.1}$$

and its generalization

$$x'' + q(t)f(x) = 0 \tag{2.2}$$

let $q(t)$ and $f(x)$ satisfy the conditions (2.3), (2.4) and (2.5) of §2.1 and assume the conclusion of Theorem 2.1. The following results of Moore and Nehari [16] can be applied to obtain further information regarding certain properties of an equation of mixed type.

Lemma 3.1 (Moore and Nehari [16])

Let $f(t,x)$ be continuous for $0 \leq t < \infty$ and $-\infty < x < \infty$ and let solutions to initial value problems for

$$x'' + f(t,x) = 0 \tag{3.1}$$

be unique. Let u, v and w be solutions to (3.1). If

$$u(t) \leq v(t) \leq w(t)$$

in an interval $[t_1,t_2]$, $0 \leq t_1 < t_2 < \infty$, and if $f(t,x)$ is a convex function of x for $\min u < x < \max w$ and any fixed $t > 0$, then

$$\phi(t) = (w-v)(v'-u') - (v-u)(w'-v') \qquad (3.2)$$

is a strictly increasing function in $[t_1,t_2]$ except when (3.1) is linear.

Proof

Making use of the fact that u, v and w are solutions of (3.1) it follows from (3.2) that

$$\phi'(t) = (v-u)[f(t,w)-f(t,v)] - (w-v)[f(t,v)-f(t,u)] \ .$$

Since solutions to initial value problems for (3.1) are unique it follows that $u < v < w$ in (t_1,t_2), and since $f(t,x)$ is convex in x

$$\frac{f(t,v)-f(t,u)}{v-u} \leq \frac{f(t,w)-f(t,v)}{w-v}$$

for t in (t_1,t_2) , with equality only if the points $(u,f(t,u))$, $(v,f(t,v))$ and $(w,f(t,w))$ lie on a straight line. If $s_1 < s_2$ lie in $[t_1,t_2]$ then $\phi'(t)$ is strictly positive in (s_1,s_2) unless $f(t,x)$ is linear in x , and therefore $\phi(s_2) > \phi(s_1)$.

Lemma 3.2

Let $f(t,x)$ satisfy the hypotheses of Lemma 3.1. If $u(t)$, $v(t)$, $w(t)$ are solutions of (3.1) such that $u(t) < w(t)$ and $v(t) < w(t)$

for t in (t_1, t_2) , $0 \leq t_1 < t_2 < \infty$, then the curves $y = u(t)$ and $y = v(t)$ cannot intersect in $[t_1, t_2]$ more than once, except when (3.1) is linear.

Proof.

Suppose there are two points $s_1 < s_2$ in $[t_1, t_2]$ such that $u(s_1) = v(s_1)$ and $u(s_2) = v(s_2)$ with $u(t) < v(t)$ for t in (s_1, s_2) . Then $v'(s_1) > u'(s_1)$ and $v'(s_2) < u'(s_2)$, hence by (3.2),

$$\phi(s_1) = [w(s_1)-v(s_1)][v'(s_1)-u'(s_1)] > 0$$

$$\phi(s_2) = [w(s_2)-v(s_2)][v'(s_2)-u'(s_2)] < 0$$

which contradicts Lemma 3.1.

It should be pointed out that the above lemmas apply to equation (2.2) under our conditions only if u, v, w are non-negative in $[t_1, t_2]$. Let us assume that

$$f(x) \text{ is convex for } x > 0 \text{ and concave for } x < 0 . \tag{3.3}$$

Then if $v(t)$ is a solution of (2.2) with $v(t_1) = v(t_2) = 0$ and $v(t) > 0$ on (t_1, t_2) and $w(t)$ is a solution of (2.2) with $w(t) > 0$ on $[t_1, t_2]$, it follows from Lemma 3.2 and the fact that $u \equiv 0$ is a solution of (2.2), that there must exist a point t_o in (t_1, t_2) such that $v(t_o) > w(t_o)$. One thus concludes that if a nonoscillatory solution $w(t)$ of (2.2) is postive for $t > T$ and $v(t)$ is an oscillatory solution of (2.2), then on any interval I to the right of T on which $v(t)$

describes a positive arch, there exists a point $t_o \in I$ such that $v(t_o) > w(t_o)$. We therefore have

Theorem 3.3

Assume (3.3) is satisfied and assume further that one of the following hypotheses H_1, H_2 is true:

H_1 : (2.16) holds for some $p > 1$.

H_2 : (2.17) holds for some $\alpha \geq 0$, and (2.18).

Then

(i) if $\int_o^\infty tq(t)dt < \infty$ then every oscillatory solution to (2.2) is unbounded.

(ii) If (2.2) admits a non-trivial solution $x(t)$ such that $\lim_{t \to \infty} x(t) = 0$ then $q \in O'$.

(iii) If all solutions of (2.2) are bounded then $q \in N'$ if and only if $\int_o^\infty tq(t)dt < \infty$.

Proof.

(i) It was shown by Atkinson [1] for equation (2.1) and by Wong [24] for equation (2.2) that the condition $\int_o^\infty tq(t)dt < \infty$ implies for any $\alpha > 0$, the existence of a solution $y_\alpha(t)$ satisfying

$$\lim_{t \to \infty} y_\alpha(t) = \alpha \quad , \quad \lim_{t \to \infty} y'_\alpha(t) = 0$$

which is clearly nonoscillatory. In view of Lemma 3.2 every oscillatory

solution $x(t)$ of (2.2) must satisfy $x(t_k) > y_\alpha(t_k)$ on an infinite sequence of points $t_k \to \infty$. Since $\alpha > 0$ is arbitrary it then follows that $\limsup_{t \to \infty} x(t) = +\infty$ for any oscillatory solution $x(t)$ of (2.2).

(ii) Since a nonoscillatory solution of (2.2) must be either eventually positive and non-decreasing or eventually negative and non-increasing, a solution $x(t)$ can satisfy $\lim_{t \to \infty} x(t) = 0$ only if $x(t)$ is oscillatory. Such a solution is necessarily bounded, and hence $\int_0^\infty tq(t)dt = +\infty$ as a result of part (i) above. But this implies that $q \in \mathcal{O}'$.

(iii) Suppose all solutions of (2.2) are bounded. If $\int_0^\infty tq(t)dt < \infty$ then no oscillatory solutions exist by part (i), whereas if $q \in \mathcal{N}'$ then the integral is finite by Theorems 2.3 and 2.4.

Theorem 3.3 (iii) provides various necessary and sufficient conditions that all solutions of (2.2) be nonoscillatory. The result is not as strong as one would wish, since it requires the knowledge that all solutions of (2.2) be bounded, a most restrictive condition on the differential equation. The theorem of Atkinson, (Theorem 2.2 above) giving sufficient conditions that all solutions of (2.1) be nonoscillatory, does not impose such a restriction and, moreover, applies even in cases when unbounded solutions exist. As an example, consider the equation

$$x'' + e^{-t}x^{2n+1} = 0 \quad . \tag{3.4}$$

It follows from Theorem 2.2 that all solutions of (3.4) are nonoscillatory. On the other hand, the following theorem of Moore and Nehari shows that (3.4) admits unbounded solutions:

Theorem 3.4 (Moore and Nehari [16])

Equation (2.1) has solutions for which

$$\lim_{t \to \infty} \frac{x(t)}{t} = \alpha > 0 \tag{3.5}$$

if and only if

$$\int_0^\infty t^{2n+1} q(t)dt < +\infty .$$

It is clear from the above that a mixed equation of the form (2.1), (i.e., one for which $q \in M$), must admit unbounded solutions. In any equation of the form (2.1), all of whose solutions are bounded, $q(t)$ belongs to class O or N according as the integral $\int_0^\infty tq(t)dt$ is infinite or finite. (The two preceeding statements also apply to equation (2.2) providing one assumes condition (3.3) and one of the hypotheses H_1, H_2.) One of the most general criteria known for boundedness of all solutions to (2.2) is the following theorem of Petty and Leitman [. The hypotheses of the theorem will include the following assumptions and notation:

(i) the function $f(x)$ is continuous on the open interval $I_1 = (-\alpha,\beta)$; $\alpha,\beta > 0$, and $xf(x) \geq 0$ for $x \in I_1$. The interval I_1 need not necessarily be bounded and α or β may be $+\infty$.

(ii) let $F(\tau) = 2\int_0^\tau f(x)dx$, $\tau \in I_1$. Because of (i), $F(\tau) \geq 0$ and we may define $F(-\alpha)$, $F(\beta)$ by limits.

(iii) we set $M = \min [F(-\alpha),F(\beta)]$, where M may be $+\infty$. Then

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

the interval $I(\gamma) = \{\tau \in I_1 \mid F(\tau) \leq \gamma\}$ is compact if $0 \leq \gamma < M$;

(iv) the functions $b(t)$, $c(t)$ are assumed to be continuous on the half open interval $I_o = [t_o, T)$ where $T > t_o$ and T may be $+\infty$. It is further assumed that $c(t)$ is positive and absolutely continuous on every compact interval contained in I_o ;

(v) the function $g(t) = \min \lfloor 2b(t), -\frac{c'(t)}{c(t)} \rfloor$ is then summable on every compact interval in I_o and $a(t) = c(t) \exp (\int_{t_o}^{t} g(s)ds)$ is non-increasing with increasing t on I_o ;

(vi) we define the constants k, K by

$$\lim_{t \to T} a(t) = k$$

$$\inf_{t \in I_o} \exp \left(\int_{t_o}^{t} g(s)ds \right) = K \ .$$

<u>Theorem 3.5</u> (Petty and Leitmann [19])

Let the above hypotheses (i)-(vi) be satisfied and let $x(t)$ be a solution to $x'' + b(t)x' + c(t)f(x) = 0$, not necessarily unique, satisfying the initial conditions $x(t_o) = x_o$, $x'(t_o) = x'_o$ where $x_o \in I_1$. If $k > 0$ and $A_o = c(t_o)F(x_o) + (x'_o)^2 < kM$, then the solution $x(t)$ may be continued to all $t \in I_o$ and $x(t)$ lies in the compact interval $I(A_o k^{-1})$ for all $t \in I_o$. In addition, if $K > 0$ then $|x'(t)| \leq (A_o K^{-1})^{1/2}$ for all $t \in I_o$.

In the special case of equation (2.2), where $q(t)$, $f(x)$ satisfy the conditions of §2.1 and $q(t)$ is absolutely continuous on every compact subinterval of $[0,\infty)$, we have

$$I_1 = (-\infty,\infty)\ ; \qquad M = +\infty\ ; \qquad I_o = [0,\infty)\ ;$$

$$g(t) = -\left(\frac{q'(t)}{q(t)}\right)_+$$

$$k = \lim_{t\to\infty} q(t)\exp\left(-\int_o^t \left(\frac{q'(s)}{q(s)}\right)_+ ds\right)$$

$$K = \inf_{t\in[0,\infty)} \exp\left(-\int_o^t \left(\frac{q'(s)}{q(s)}\right)_+\right) ds$$

with

$$F(\tau) = 2\int_o^\tau f(x)dx\ , \qquad \tau \in (-\infty,\infty)$$

and

$$I(\gamma) = \{\tau \mid F(\tau) \le \gamma\}\ .$$

Theorem 3.5 then asserts that if $k > 0$ then the solution $x(t)$ of (2.2) with initial values $x(0) = a$, $x'(0) = b$, lies in the compact interval $I(Ak^{-1})$ for all $t \ge 0$, where $A = q(0)F(a) + b^2$. In addition, if $K > 0$ then $|x'(t)| \le (AK^{-1})^{1/2}$ for all $t \ge 0$.

Now, suppose the hypotheses of Theorem 3.3 are true. Then equation (2.2) can be of mixed type only if all oscillatory solutions are unbounded. A necessary condition for the existence of unbounded solutions is that the constant k in Theorem 3.5 satisfy $k = 0$, which implies that either

$$\lim_{t\to\infty} q(t) = 0 \tag{3.7}$$

or

$$\int_o^\infty \left(\frac{q'(s)}{q(s)}\right)_+ ds = +\infty\ . \tag{3.8}$$

We have therefore proved:

Theorem 3.6

A necessary condition that $q \in M$ is that either (3.7) or (3.8) hold. If $f(x)$ satisfies the hypotheses of Theorem 3.3 then a necessary condition that $q \in M'$ is that either (3.7) or (3.8) hold.

The following known results are consequences of Theorem 3.5.

Theorem 3.7 (Utz [22])

If $q(t)$ is differentiable and $q'(t) \leq 0$ for $t \geq T$ with $\lim_{t \to \infty} q(t) > 0$ then all solutions of (2.1) are bounded (and, in fact, oscillatory).

Theorem 3.8 (Waltman [23])

If $q(t)$ is differentiable and $q'(t) \geq 0$ for $t \geq T$ then all solutions of (2.1) are bounded (and, in fact, oscillatory).

In addition to the results of Theorem 3.6, it is also known that

$$\int_0^\infty tq(t)dt < \infty \tag{3.9}$$

is a necessary condition that $q \in M$ (or that $q \in M'$ if $f(x)$ satisfies the conditions of Theorem 3.3). Moreover, if q is differentiable and $q'(t) \leq 0$ for $t \geq T$ then

$$\int_0^\infty t^{2n+1} q(t)dt = + \infty \tag{3.10}$$

is a necessary condition for $q \in M$. (For conditions analogous to (3.10) pertaining to equation (2.2) see Theorem 2 of Macki-Wong [15]).

It is seen, therefore, that a considerable number of restrictions must be placed on $q(t)$ in order that it might belong to class M or M'.

We conclude this section with an additional comment on non-oscillatory solutions of (2.1).

It is well known that the nonoscillatory character of a non-linear equation of the form (2.1) is something very different from the corresponding property for a linear equation. In fact,

Theorem 3.9 (Moore and Nehari [16])

For any t_1, t_2 such that $0 < t_1 < t_2 < \infty$, and for any non-negative integer m, there exists a solution of (2.1) which vanishes at $t = t_1$ and $t = t_2$ and has precisely m zeros in (t_1, t_2).

It should be remarked that if $q \in M$ the solution exhibited by Theorem 3.9 may very well be an oscillatory solution, as is the case for equation (2.28) discussed earlier, which has the property that a solution with one zero must oscillate. However, even if $q \in N$, solutions with arbitrarily large numbers of zeros will exist, by Theorem 3.9. The situation exhibited by equation (2.28), wherein a solution having one zero is oscillatory, indicates that we can separate the class M into two subclasses characterized by the conditions (i) nonoscillatory solutions are free of zeros, and (ii) there exist non-oscillatory solutions having a finite positive number of zeros. Although

equation (2.28) provides an example of a function $q(t)$ belonging to the subclass determined by (i), it is not presently known whether or not the class (ii) is empty.

§3.2 Dependence on initial conditions of the number of zeros of a solution on an interval.

Returning now to a consideration of the second question posed in Chapter I, that is, an investigation of the manner in which the behaviour of solutions is dependent upon initial conditions, consider again the equation

$$x'' + q(t)f(x) = 0 , \qquad q(t) > 0 , \tag{2.2}$$

where $q(t)$ is continuous on $[0,\infty)$, $f(x)$ is continuous for $x \in (-\infty,\infty)$, $xf(x) > 0$ for $x \neq 0$. It is assumed throughout this section that for any $t_o \geq 0$, $a \in R$, $b \in R$, the solution of (2.2) satisfying

$$x(t_o) = a , \qquad x'(t_o) = b$$

is unique and exists on $[0,\infty)$.

For the purposes of this section we may, without loss of generality, restrict our attention to the case in which the initial value a is non-negative. Indeed, if a solution $x(t)$ of (2.2) satisfies initial conditions

$$x(t_o) = a < 0 , \qquad x'(t_o) = b$$

then the function $y(t) = -x(t)$ is a solution of

$$y'' + q(t)g(y) = 0$$

$$y(t_o) = -a > 0 \ , \qquad y'(t_o) = -b$$

where $g(u) = -f(-u)$ and satisfies all the assumptions made for $f(x)$ at the beginning of this section.

Lemma 3.10

Assume that

$$\frac{f(x)}{x} \text{ is non-decreasing for } x > 0 \text{ and non-increasing for } x < 0 \ , \tag{3.11}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = +\infty \ , \tag{3.12}$$

and let $0 < T < \infty$. Given $t_o \in (0,T)$ and $R_1 > 0$, there exists $R_o = R_o(R_1, t_o)$ such if $0 \not\equiv x(t)$ solves (2.2) and at some $T_o \in [0, t_o)$ we have $\rho_o = \rho(T_o) = [x(T_o)^2 + x'(T_o)^2]^{1/2} > R_o$, then $x(t)$ has at least one zero in (T_o, T), and if $T_1 \in (T_o, T)$ is the first such zero of $x(t)$ then $|x'(T_1)| > R_1$.

Theorem 3.11

Assume the hypotheses of Lemma 3.10. Given $B > 0$ and a positive integer m, there exists $A = A(m,B)$ such that for any a, b satisfying

$$\rho_o = (a^2 + b^2)^{1/2} > A \ ,$$

the solution $x(t)$ of equation (2.2) for which

$$x(0) = a \ , \qquad x'(0) = b$$

will have at least m zeros on $(0,B)$. (We use an argument suggested by the proof of an unpublished theorem of J. Mallet-Paret.)

Proof of Lemma 3.10

As was pointed out above, we may assume that $x(T_o) \geq 0$. Since $q(t)$ is positive and continuous, there exist constants α, β such that $0 < \alpha < q(t) < \beta$ for $t \in [0,T]$. Define the sets

$$Q_1 = \{(u,v)/u > 0, \ v \geq 0\}$$

$$Q_2 = \{(u,v)/u \leq 0, \ v > 0\} \ .$$

Since we have assumed $x(T_o) \geq 0$ and have discarded the trivial initial conditions, then either $x(T_o) = 0$, $x'(T_o) < 0$, or the point $(x'(T_o),x(T_o))$ lies in the subset $M = Q_1 \cup Q_2$ of the real plane. If $x(T_o) = 0$, $x'(T_o) < 0$ then we may, as above, consider the function $y(t) = -x(t)$ for which $(y'(T_o),y(T_o))$ does lie in the set M . Thus we need prove the lemma only for $(x'(T_o),x(T_o))$ belonging to M , that is, lying in the open upper half plane plus the positive semi-axis.

Define the functions $\rho(t) > 0$ and $\theta(t)$ implicitly by

$$\left.\begin{aligned} x(t) &= \rho(t) \sin \theta(t) \\ x'(t) &= \rho(t) \cos \theta(t) \end{aligned}\right\} \qquad (3.13)$$

where $\theta_o = \theta(T_o) = \arctan \dfrac{x(T_o)}{x'(T_o)}$ is prescribed so that

$$0 \leq \theta_o < \frac{\pi}{2} \qquad \text{if} \qquad (x'(T_o),x(T_o)) \in Q_1$$

$$\frac{\pi}{2} \leq \theta_o < \pi \qquad \text{if} \qquad (x'(T_o),x(T_o)) \in Q_2$$

and $\theta(t)$ is defined so as to be continuous for $t \geq T_o$. Differentiating each of the equalities (3.13) and making use of (3.13) and (2.2) gives

$$\left.\begin{aligned} \theta'(t) &= q(t)\,\frac{\sin\theta(t)}{\rho(t)}\,f(x(t)) + \cos^2\theta(t) \\ \rho'(t) &= (x(t) - q(t))f(x(t))\cos\theta(t)\ . \end{aligned}\right\} \tag{3.14}$$

Since $\rho(t) > 0$ and $uf(u) > 0$ for $u \neq 0$ it follows from (3.14) that $\theta'(t) \geq 0$ for all t, for

$$q\,\frac{\sin\theta}{\rho}\,f(x) + \cos^2\theta \geq q\,\frac{\sin\theta}{\rho}\,f(x) = \frac{q}{\rho^2}\,xf(x) \geq 0\ .$$

Define

$$\left.\begin{aligned} \rho_1(t) &= (x'(t))^2 + 2\alpha\int_o^{x(t)} f(u)\,du \\ \rho_2(t) &= (x'(t))^2 + 2\beta\int_o^{x(t)} f(u)\,du \end{aligned}\right\}\ . \tag{3.15}$$

Then

$$\left.\begin{aligned} \rho_1'(t) &= 2x'f(x)(\alpha-q) \\ \rho_2'(t) &= 2x'f(x)(\beta-q) \end{aligned}\right\}\ . \tag{3.16}$$

Therefore

$$\begin{aligned} &\rho_1'(t) \geq 0 \ \text{ in } \ Q_2\ , \qquad \rho_1'(t) \leq 0 \ \text{ in } \ Q_1 \\ &\rho_2'(t) \geq 0 \ \text{ in } \ Q_1\ , \qquad \rho_2'(t) \leq 0 \ \text{ in } \ Q_2 \end{aligned}$$

and, on $Q_1 \cup Q_2$ we have $\rho_1'(t)$ and $\rho_2'(t)$ are non-zero except when either $x(t) = 0$ or $x'(t) = 0$.

Case (i) <u>Initial values lying in</u> Q_1

We then have $0 \le \theta_o < \frac{\pi}{2}$. As long as $(x'(t),x(t))$ remains in Q_1 , then $\theta_o \le \theta(t) < \frac{\pi}{2}$ and $\sin\theta(t) \ge 0$. From (3.14) $\theta'(t) \ge \cos^2\theta(t)$ and thus

$$\int_{\theta_o}^{\theta(t)} \frac{d\phi}{\cos^2\phi} \ge t - T_o \tag{3.17}$$

as long as $\theta_o \le \theta(t) < \frac{\pi}{2}$. Thus

$$\tan\theta(t) - \tan\theta_o \ge t - T_o \ . \tag{3.18}$$

Choose θ_1 in $(\theta_o, \frac{\pi}{2})$ such that

$$\tan\theta_1 - \tan\theta_o < \frac{t_o - T_o}{4} \ .$$

By (3.18) the function $\theta(t)$ takes on the value θ_1 at some point $t_1 > T_o$. Then

$$\tan\theta(t_1) - \tan\theta_o \ge t_1 - T_o$$

so t_1 must satisfy

$$t_1 - T_o \le \frac{t_o - T_o}{4} \ . \tag{3.19}$$

Now

$$\rho_1'(t) \le 0 \text{ in } Q_1 \qquad \text{and} \qquad \rho_1'(t) = 0 \text{ in } Q_1$$

if and only if $x(t) = 0$. Thus for $t \ge T_o$

$$\rho_1(t) \le \rho_1(T_o) = (x'(T_o))^2 + 2\alpha\int_0^{x(T_o)} f(u)du$$

as long as $(x'(t),x(t))$ remains in Q_1. Let

$$K_1 > \begin{cases} 1 \; ; & \text{if } \; x(T_o) = 0 \; ; \\ \max\left\{1, \dfrac{2\alpha}{x(T_o)^2} \displaystyle\int_o^{x(T_o)} f(u)du\right\} \; ; & \text{if } \; x(T_o) > 0 \; . \end{cases}$$

Then

$$\rho_1(t) \leq \rho_1(T_o) < K_1\rho_o^2 \quad \text{if} \quad (x'(t),x(t)) \in Q_1 \; .$$

Similarly, $\rho_2'(t) \geq 0$ in Q_1 and $\rho_2'(t) = 0$ in Q_1 if and only if $x(t) = 0$. Thus for $t \geq T_o$

$$\rho_2(t) \geq \rho_2(T_o) = (x'(T_o))^2 + 2\beta \int_o^{x(T_o)} f(u)du$$

as long as $(x'(t),x(t))$ remains in Q_1. Let

$$0 < K_2 < \begin{cases} 1 \; ; & \text{if } \; x(T_o) = 0 \; ; \\ \min\left\{1, \dfrac{2\beta}{x(T_o)^2} \displaystyle\int_o^{x(T_o)} f(u)du\right\} \; ; & \text{if } \; x(T_o) > 0 \; . \end{cases}$$

Then

$$\rho_2(t) \geq \rho_2(T_o) > K_2\rho_o^2 \quad \text{if} \quad (x'(t),x(t)) \in Q_1 \; .$$

Continuing from our observation that $\theta(t)$ takes on the value θ_1 at some point $t_1 > T_o$, in which case (3.19) holds, we let $x_1 = x(t_1)$ and $x'(t_1) = \rho(t_1)\cos\theta_1 = \rho(t_1)\sin\theta_1 \dfrac{\cos\theta_1}{\sin\theta_1} = \dfrac{x_1}{\tan\theta_1}$. Then

$$\rho_2(t_1) = \frac{x_1^2}{\tan^2\theta_1} + 2\beta \int_o^{x_1} f(u)du > K_2\rho_o^2 \; . \tag{3.20}$$

But the function

$$\frac{u^2}{\tan^2\theta_1} + 2\,\beta \int_o^u f(\sigma)d\sigma$$

is an increasing function of u, and thus it follows from (3.20) that $x_1 \geq \gamma > 0$ where γ solves

$$\frac{\gamma^2}{\tan^2\theta_1} + 2\beta \int^{\gamma} f(\sigma)d\sigma = K_2\rho_o^2$$

and $\gamma = \gamma(\rho_o)$ satisfies $\lim_{\rho_o \to \infty} \gamma(\rho_o) = +\infty$. Since $x'(t) > 0$ in Q_1 we have $x(t) \geq x_1 \geq \gamma > 0$ for $t \geq t_1$, $(x'(t),x(t)) \in Q_1$. Moreover, in Q_1 we have from (3.13) $\rho(t) = \dfrac{x(t)}{\sin\theta(t)}$ for $t \geq t_1$ and thus, from (3.13) follows

$$\theta'(t) \geq q(t)\sin^2\theta\,\frac{f(x(t))}{x(t)} \geq \frac{\alpha f(x_1)}{x_1}\sin^2\theta \quad .$$

Therefore

$$\int_{\theta_1}^{\theta(t)} \frac{d\phi}{\sin^2\phi} \geq \frac{\alpha f(x_1)}{x_1}(t-t_1)\ ,$$

$$-\cot\theta + \cot\theta_1 \geq \frac{\alpha f(x_1)}{x_1}(t-t_1)\ ,$$

$$-\cot\theta \geq \frac{\alpha f(x_1)}{x_1}(t-t_1) - \cot\theta_1$$

for $t_1 \leq t \leq T$ and $\theta_1 \leq \theta(t) \leq \frac{\pi}{2}$. Thus it cannot be true that $\theta(T_o + \frac{t_o - T_o}{2}) \leq \frac{\pi}{2}$ for all ρ_o. Indeed, if $\theta(T_o + \frac{t_o - T_o}{2}) \leq \frac{\pi}{2}$ for all ρ_o then

$$0 \geq -\cot\theta\left(T_o + \frac{t_o - T_o}{2}\right) \geq \frac{\alpha f(x_1)}{x_1}\left(T_o + \frac{t_o - T_o}{2} - t_1\right) - \cot\theta_1 \geq$$

$$\geq \frac{\alpha f(x_1)}{x_1}\left(\frac{t_o - T_o}{2} - (t_1 - T_o)\right) - \cot\theta_o .$$

But $-(t_1 - T_o) \geq -\left(\frac{t_o - T_o}{4}\right)$. Thus

$$0 \geq \frac{\alpha f(x_1)}{x_1}\left(\frac{t_o - T_o}{4}\right) - \cot\theta_o . \tag{3.21}$$

But $x_1 \geq \gamma > 0$ and $\lim_{\rho_o \to \infty} \gamma(\rho_o) = +\infty$. Thus $x_1 \to \infty$ as $\rho_o \to \infty$ and hence $\frac{f(x_1)}{x_1} \to \infty$, which gives a contradiction of (3.21). Thus there exists $N = N(t_o)$ such that if $\rho_o > N$ then $\theta\left(T_o + \frac{t_o - T_o}{2}\right) > \frac{\pi}{2}$. Now let t_2 be chosen such that $T_o < t_1 < t_2 < T_o + \frac{t_o - T_o}{2} < T$ with with $\theta(t_2) = \frac{\pi}{2}$. Let $x(t_2) = \eta > 0$. Since $x'(t) > 0$ in Q_1 for $T_o < t < t_2$, then $x(t_2) \geq x(t_1) = x_1 \geq \gamma$ and hence $\eta \geq \gamma$. Therefore

$$\lim_{\rho_o \to \infty} \eta(\rho_o) = +\infty . \tag{3.22}$$

We have show that for a solution $x(t)$ having initial conditions $(x'(T_o), x(T_o))$ lying in the first quadrant Q_1, it is possible to force $(x'(t), x(t))$ into the second quadrant Q_2 within an interval $[T_o, T_o + \frac{t_o - T_o}{2}]$ by taking $\rho_o^2 = x'(T_o)^2 + x(T_o)^2$ sufficiently large, and this can be done in such a way as to make $|x(t_2)|$ arbitrarily large at the first zero, t_2, of $x'(t)$. There remains to show that solutions with sufficiently large initial conditions can be forced into the third quadrant within the t interval (T_o, t_o) in such a way as will result in $|x'(T_1)|$ being arbitrarily large at the first zero, T_1, of $x(t)$ in (T_o, t_o); which is, clearly, equivalent to proving the

lemma under case (ii), that is, the case in which the initial values belong to the set Q_2 .

Choose θ_3 in $\frac{\pi}{2} < \theta_3 < \pi$ such that $\tan\theta_3 > -\frac{(t_o - T_o)}{4}$. Then, as long as $|x(t)| \geq \delta > 0$ we have from (3.14)

$$\theta' = q \sin^2\theta \frac{f(x)}{x} + \cos^2\theta \geq \frac{\alpha f(x)}{x} \sin^2\theta \quad .$$

For $(x'(t),x(t)) \in Q_2$, $x'(t) \leq 0$ so $x(t)$ is non-increasing. Thus $0 < \delta \leq x(t) \leq x(t_2)$ in Q_2 as long as $t < t^*$ where $x(t^*) = \delta$. Therefore $\frac{f(x(t_2))}{x(t_2)} \geq \frac{f(x(t))}{x(t)} \geq \frac{f(\delta)}{\delta} > 0$ as long as $x(t) \geq \delta$,

$$\theta'(t) \geq \frac{\alpha f(\delta)}{\delta} \sin^2\theta(t) \; , \quad \text{for} \quad x(t) \geq \delta \; .$$

Now $\theta_2 = \theta(t_2) = \frac{\pi}{2}$ and thus

$$\int_{\theta_2}^{\theta(t)} \frac{d\phi}{\sin^2\phi} \geq \int_{t_2}^{t} \frac{\alpha f(\delta)}{\delta} ds = \frac{\alpha f(\delta)}{\delta} (t-t_2)$$

or

$$- \cot\theta(t) \geq \frac{\alpha f(\delta)}{\delta} (t-t_2)$$

as long as $\frac{\pi}{2} < \theta(t) < \pi$, $x(t) \geq \delta$. Suppose $\theta(t_2 + \frac{t_o - T_o}{4}) \leq \theta_3 < \pi$ for any choice of $x'(T_o) \geq 0$. Since $t_2 < T_o + \frac{t_o - T_o}{2} < t_o < T$ we have $t_2 + \frac{t_o - T_o}{4} < T_o + \frac{3(t_o - T_o)}{4} < t_o < T$ and, since $\frac{\pi}{2} \leq \theta(t_2 + \frac{t_o - T_o}{4}) \leq \theta_3 < \pi$ then $x(t_2) \geq x(t) \geq x(t_2 + \frac{t_o - T_o}{4}) > 0$ for t in the initerval $I = [t_2,\tau]$, $\tau = t_2 + \frac{t_o - T_o}{4}$. Thus

$$\theta' \geq \frac{\alpha f[x(\tau)]}{x(\tau)} \sin^2\theta \, , \qquad t \in I \, ,$$

from which follows

$$-\cot\theta(\tau) \geq \frac{\alpha f[x(\tau)]}{x(\tau)}\left(\frac{t_o - T_o}{4}\right) \quad .$$

Define

$$q_1(t) = \begin{cases} q(t) \, ; & 0 \leq t \leq T \\ q(T) \, ; & T \leq t \quad . \end{cases}$$

Then all solutions of

$$y'' + q_1(t)f(y) = 0 \tag{3.23}$$

are oscillatory beyond T . (Theorem 2.2). Let $\overline{x}(t)$ be the solution of (3.23) satisfying $\overline{x}(T_o) = x(T_o)$, $\overline{x}'(T_o) = x'(T_o)$. Then $\overline{x}(t) \equiv x(t)$ for t in $[0,T)$ and $\overline{x}(t)$ is oscillatory beyond T . Let $\overline{\rho}(t)$, $\overline{\theta}(t)$, $\overline{\rho}_1(t)$ and $\overline{\rho}_2(t)$ respectively be defined for $\overline{x}(t)$ by (3.13) and (3.15). Since $\overline{x}(t)$ is oscillatory, there exists some $t_3 > t_2$ such that $\overline{\theta}(t_3) = \theta_3$. (It might be true, however, that $t_3 > T$. It is shown below that for sufficiently large $\overline{x}'(T_o)$, t_3 is, in fact, less than t_o .) Since $\overline{\theta}(t)$ is continuous and increasing in $[t_2,t_3]$ and $\overline{x}'(t) < 0$ in $[t_2,t_3]$ we have $\overline{x}(t_2) \geq \overline{x}(\tau) \geq \overline{x}(t_3)$. Now $\overline{x}'(t_3) = \dfrac{\overline{x}(t_3)}{\tan\theta_3}$ so it follows from the definitions (3.15) (with ρ_1, x replaced by $\overline{\rho}_1$, $\overline{x}$) that

$$\overline{\rho}_1(t_3) = \frac{\overline{x}^2(t_3)}{\tan^2\theta_3} + 2\alpha \int_o^{\overline{x}(t_3)} f(u)du \quad .$$

But $\bar{\rho}_1' \geq 0$ in Q_2 , so $\bar{\rho}_1(t_3) \geq \bar{\rho}_1(t_2)$, hence

$$\bar{\rho}_1(t_3) = \frac{\bar{x}^2(t_3)}{\tan^2\theta_3} + 2\alpha \int_0^{\bar{x}(t_3)} f(u)du \geq \bar{\rho}_1(t_2) = 2\alpha \int_0^{\eta} f(u)du$$

since $\eta = x(t_2) = \bar{x}(t_2)$ because $\bar{x}(t) \equiv x(t)$ on $[0,T)$. As $\rho_o \to \infty$, $\eta \to \infty$ and hence $\bar{\rho}_1(t_3) \to \infty$, which implies $\lim_{\rho_o \to \infty} \bar{x}(t_3) = \infty$. Since $\bar{x}(t_3) \leq \bar{x}(\tau)$ it follows that $\lim_{\rho_o \to \infty} \bar{x}(\tau) = \infty$. But

$$- \cot \bar{\theta}(\tau) \geq \frac{\alpha f[\bar{x}(\tau)]}{\bar{x}(\tau)} \left(\frac{t_o - T_o}{4}\right)$$

and therefore $\cot \bar{\theta}(\tau) \to -\infty$ as $\rho_o \to \infty$, hence

$$\bar{\theta}(\tau) \to \pi \quad \text{as} \quad \rho_o \to \infty . \tag{3.24}$$

But $\tau = t_2 + \frac{t_o - T_o}{4} < T_o + \frac{3(t_o - T_o)}{4}$ and thus $\bar{\theta}(t) \equiv \theta(t)$ at $t_2 + \frac{t_o - T_o}{4}$. Thus (3.24) is a contradiction of the supposition that $\theta(t_2 + \frac{t_o - T_o}{4}) \leq \theta_3 < \pi$ for any choice of $x'(T_o) \geq 0$. Therefore for any $\theta_3 < \pi$, it is possible to choose $x'(T_o)$ so large that

$$\theta(t_2 + \frac{t_o - T_o}{4}) > \theta_3 = \bar{\theta}(t_3) = \theta(t_3) .$$

Thus

$$t_3 < t_2 + \frac{t_o - T_o}{4} < T_o + \frac{3(t_o - T_o)}{4}$$

and $\bar{x}(t) \equiv x(t)$ on $[0,t_3)$. Now for $\theta_3 \leq \bar{\theta} < \frac{3\pi}{2}$ we have $\bar{\theta}' \geq \cos^2 \bar{\theta}$ since $\sin \bar{\theta}(t)$ and $f(\bar{x}(t))$ have the same algebraic sign. Therefore

$$\int_{\theta_3}^{\overline{\theta}(t)} \frac{d\phi}{\cos^2\phi} \geq t - t_3 \ .$$

Let T_1 be chosen such that $\overline{\theta}(T_1) = \pi$. (This is possible because $\overline{x}(t)$ is oscillatory. It may happen that $T_1 > T$ but this will be ruled out in what follows.) Then

$$\tan \overline{\theta}(T_1) - \tan \overline{\theta}(t_3) = - \tan \theta_3 \geq T_1 - t_3$$

$$- \tan \theta_3 > T_1 - (T_o + \frac{3(t_o - T_o)}{4}) \ .$$

But $\tan \theta_3 > - \frac{(t_o - T_o)}{4}$ and therefore

$$T_1 - (T_o + \frac{3(t_o - T_o)}{4}) < \frac{t_o - T_o}{4}$$

$$T_1 < t_o \ .$$

But $\overline{\theta}(t) \equiv \theta(t)$ on $[0,T]$ and therefore for sufficiently large ρ_o we have $\theta(t_o) > \pi$ and $x(t)$ has a zero at T_1 in (T_o, t_o), since $\theta(T_1) = 0$.

In Q_2, ρ_1 increases so $\rho_1(T_1) \geq \rho_1(t_2)$. Therefore

$$x'^2(T_1) \geq x'^2(t_2) + 2\alpha \int_o^{\eta} f(u)du \geq 2\alpha \int_o^{\eta} f(u)du \ .$$

But $\eta \to \infty$ as $\rho_o \to \infty$ and therefore so does $|x'(T_1)|$. Thus, there exists $R_o = R_o(R_1, t_o)$ such that if $\rho_o > R_o$ then $x(t)$ has a zero in (T_o, t_o) and $|x'(T_1)| > R_1$ if T_1 is the first zero of $x(t)$ in (T_o, t_o). The proof of Lemma 3.10 is complete.

Proof of Theorem 3.11

Let R_m be a given positive number. Making use of Lemma 3.10, there exists $R_{m-1} = R_{m-1}(R_m, B)$ such that if $x(t)$ solves (2.2) and at some point $\tau \in (0, B - \frac{1}{m+1})$ we have $[x(\tau)^2 + x'(\tau)^2)]^{1/2} > R_{m-1}$ then there exists a point $T_m \in (\tau, B)$ such that $x(T_m) = 0$ and $|x'(T_m)| > R_m$. Suppose $R_m, R_{m-1}, \ldots, R_{m-k+1}$ have been chosen for some k, $1 \leq k < m-1$. Then, by Lemma 3.10 there exists $R_{m-k} = R_{m-k}(R_{m-k+1}, B)$ such that if $x(t)$ solves (2.2) and at some point $\tau \in (0, \frac{m-k}{m+1} B)$ we have $[x(\tau)^2 + x'(\tau)^2]^{1/2} > R_{m-k}$ then there exists a point $T_{m-k+1} \in (\tau, \frac{m-k}{m+1} B)$ such that $x(T_{m-k+1}) = 0$ and $|x'(T_{m-k+1})| > R_{m-k+1}$. Let the numbers $\{R_{m-j}\}$ $j = 1,2,\ldots,m-1$ be so chosen. Then again by Lemma 3.10 there exists a number $A = A(R_1, B)$ such that if $x(t)$ solves (2.2) and $[x(0)^2 + x'(0)^2]^{1/2} > A$ there exists a point T_1 in $(0, \frac{B}{m+1})$ such that $x(T_1) = 0$. Now, let the solution $x(t)$ of (2.2) satisfy $[x(0)^2 + x'(0)^2]^{1/2} > A$, and let $T_o = 0$. Then there exists a point $T_1 \in (0, \frac{B}{m+1})$ such that $x(T_1) = 0$ and $|x'(T_1)| > R_1$. But $[x(T_1)^2 + x'(T_1)^2]^{1/2} \geq x'(T_1) > R_1$ so there exists a point $T_2 \in (T_1, \frac{2B}{m+1})$ such that $x(T_2) = 0$ and $|x'(T_2)| > R_2$. Continuing in this manner, suppose, for some $0 \leq k < m$, we have points T_j, $j = 0,1,\ldots,k$, with $T_j \in (T_{j-1}, \frac{jB}{m+1})$ such that $x(T_j) = 0$ and $|x'(T_j)| > R_j$, $j = 0,1,\ldots,k$. Then, since $[x(T_k)^2 + x'(T_k)^2]^{1/2} > R_k$, it follows that there exists a point $T_{k+1} \in (T_k, \frac{k+1}{m+1} B)$ such that $x(T_{k+1}) = 0$ and $|x'(T_{k+1})| > R_{k+1}$. Therefore $x(t)$ has a squence of m distinct zeros $T_1, T_2, \ldots, T_m$ in $(0,B)$ which completes the proof of Theorem 3.11.

We should point out that subject only to the assumptions of this

section and the hypotheses of Lemma 3.10, the conclusions of Lemma 3.10 and Theorem 3.11 are valid regardless of whether the equation (2.2) is oscillatory, mixed, or nonoscillatory. In the event that equation (2.2) is mixed, however, we know of no criterion by which one might rule out the possibility that solutions having arbitrarily large numbers of zeros on an interval $[0,B)$, $B < \infty$, may, in fact, be oscillatory on $[0,\infty)$.

CHAPTER IV

§4.1 On initial conditions which generate non-vanishing solutions

Throughout this section we again consider the equation

$$x'' + q(t)f(x) = 0 \ , \qquad q(t) > 0 \ , \tag{4.1}$$

where $q(t)$ is continuous on $[0,\infty)$, $f(x)$ is continuous for $x \in (-\infty,\infty)$, $xf(x) > 0$ for $x \neq 0$. It is also assumed that for any $t_o \geq 0$, $a \in R$, $b \in R$, the solution of (4.1) satisfying

$$x(t_o) = a \ , \qquad x'(t_o) = b \tag{4.2}$$

is unique and exists on $[0,\infty)$.

We shall, on occasion, refer to the fact that if $a < 0$ in (4.2) then the function $y(t) = -x(t)$ is a solution of

$$\left.\begin{aligned} &y'' + q(t)g(y) = 0 \\ &y(0) = -a, \quad y'(0) = -b \end{aligned}\right\} \tag{4.3}$$

where $g(u) = -f(-u)$ also satisfies the above assumptions made for $f(x)$, and for which the initial value $y(0)$ is positive.

We shall also, on occasion, wish to make one or more of the following assumptions on $f(x)$,

$$\begin{aligned} &\frac{f(x)}{x} \text{ is non-decreasing for } x > 0 \quad \text{and} \\ &\text{non-increasing for } x > 0 \ , \end{aligned} \tag{4.4}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = +\infty \ . \tag{4.5}$$

For each pair of real numbers a, b, we define the (extended) real number $N(a,b)$ to be the number of zeros on $[0,\infty)$ of the solution $x(t;a,b)$ of (4.1)-(4.2) for $t_o = 0$. For each non-negative integer k, we then define

$$S_k(a) = \{b/N(a,b) \leq k\} \quad . \tag{4.6}$$

Clearly $S_o(0) = \phi$ and $S_k(a) \subseteq S_{k+1}(a)$ for $k = 0,1,2,\ldots$.

Theorem 4.1

Let (4.4) and (4.5) be satisfied. Then

(i) $0 \notin S_o(a)$ and $S_o(a)$ is closed.

(ii) $0 \notin S_k(0)$ and $\{0\} \cup S_k(0)$ is closed for all k .

(iii) If $a \neq 0$ then $S_k(a)$ is closed for all k .

Proof:

The assertion (i) is clearly true for $a = 0$ since $S_o(0) = \phi$. Let $a > 0$ and assume $S_o(a) \neq \phi$. Suppose $x(t)$ solves (4.1) and satisfies $x(0) = a$, $x'(0) = 0$. From (4.1) follows

$$x'(t) = -\int_o^t q(s)f(x(s))ds \;, \quad t \geq 0 \; . \tag{4.7}$$

If $x(t) > 0$ for all t then $x'(t) < 0$ and $x''(t) < 0$ for $t > 0$. Thus the graph of $x(t)$ is convex downwards with negative slope to the right of zero, which is a contradiction. Therefore $x(t)$ must have a zero at some finite value $t_1 > 0$, proving that $0 \notin S_o(a)$ for $a > 0$. The proof for $a < 0$ follows from the above argument by considering

$y(t) = -x(t)$ and the observations concerning equation (4.3) above.

That $S_o(a)$ is closed for $a \neq 0$ will follow from our proof of (iii).

For the proof of (ii) we first observe that since the initial conditions $x(0) = 0 = x'(0)$ generate the identically zero solution of (4.1), then clearly $0 \notin S_k(0)$. Moreover, it follows from Theorem 3.11 that $S_k(0)$ is bounded above and below, since the solution $x(t)$ of (4.1) satisfying $x(0) = 0$, $x'(0) = b$ can be forced to vanish arbitrarily often on any compact set simply by choosing $|b|$ sufficiently large. Finally we may assume $S_k(0)$ is non-empty otherwise there is nothing to prove.

Let $\{b_j\}_{j=1}^{\infty}$ be a sequence in $S_k(0)$ converging to a point b , which is necessarily finite. We wish to show that either $b = 0$ or b belongs to $S_k(0)$.

Suppose $b \neq 0$. Let $\{x_j(t)\}_{j=1}^{\infty}$ be a sequence of solutions to (4.1) satisfying $x_j(0) = 0$, $x_j'(0) = b_j$ and let $x(t)$ be the solution satisfying $x(0) = 0$, $x'(0) = b$. We must show $x(t)$ has not more than k zeros on $[0,\infty)$.

Suppose $x(t)$ has more than k zeros on $[0,\infty)$ and let

$$0 = t_1 < t_2 < t_3 < \dots < t_k < t_{k+1}$$

be the first $k+1$ zeros of $x(t)$. Because solutions to (4.1) are unique, $x(t)$ changes sign at each of the points $t_2, t_3, \dots, t_{k+1}$. Choose $T > t_{k+1}$ such that $x'(t)$ is of one sign on $(t_{k+1},T]$. Then $x(t)$

has exactly k sign changes on $(0,T]$.

Using a standard theorem (Theorem 3.2, page 14 of Hartman [10]), we can extract subsequences $\{b_{j_n}\}$ and $\{x_{j_n}(t)\}$ such that $\{x_{j_n}(t)\}$ converges uniformly to $x(t)$ on $[0,T]$. Let

$$|x(C_i)| = \max_{t \in [t_i, t_{i+1}]} |x(t)| , \qquad i = 1,2,\ldots,k ,$$

and choose

$$\alpha = \min \{ \min_{1 \leq i \leq k} |x(C_i)| , |x(T)| \} .$$

There exists a positive integer N such that if $n \geq N$, $|x(t)-x_{j_n}(t)| < \frac{\alpha}{4}$ for all $t \in [0,T]$. In particular $|x(C_i)-x_{j_N}(C_i)| < \frac{\alpha}{4}$ for all $i = 1,2,3,\ldots,k$, and $|x(T)-x_{j_N}(T)| < \frac{\alpha}{4}$. Therefore

$$\text{sign } x_{j_N}(t) = \text{sign } x(t)$$

at least once on each of the intervals (t_i, t_{i+1}) , $i = 1,2,\ldots,k$, and $(t_{k+1}, T]$, and hence $x_{j_N}(t)$ has at least k sign changes on $(0,T]$, and thus has at least $k+1$ zeros on $[0,T]$, (recall $x_{j_N}(0) = 0$) . But this contradicts the fact that $b_{j_N} \in S_k(0)$, and therefore $x(t)$ has not more than k zeros on $[0,\infty)$. Therefore either $b = 0$ or $b \in S_k(0)$ so $\{0\} \cup S_k(0)$ is a closed set.

In proving (iii) we may assume that $a > 0$. Again it follows from Theorem 3.11 that $S_k(a)$ is bounded above and below. Let $\{b_j\}_{j=1}^{\infty}$ be a sequence in $S_k(a)$ which converges to the (finite) point b . Let $\{x_j(t)\}_{j=1}^{\infty}$ and $x(t)$ be solutions to (4.1) satisfying

$$x_j(0) = a \ , \qquad x_j'(0) = b_j$$

$$x(0) = a \ , \qquad x'(0) = b \ .$$

We must show that $x(t)$ has not more than k zeros on $[0,\infty)$. Suppose $x(t)$ has more than k zeros, and let

$$0 < t_1 < t_2 < \ldots < t_k < t_{k+1}$$

be the first $k+1$ zeros of $x(t)$. Choosing $T > t_{k+1}$ such that $x'(t)$ is of one sign on $(t_{k+1},T]$ and proceeding as in the proof of (ii) above, one arrives at the existence of a positive integer N such that $x_{j_N}(t)$ has at least $k+1$ zeros on $[0,T]$, which is a contradiction. Thus $b \in S_k(a)$ proving that $S_k(a)$ is closed, which completes the proof.

An immediate consequence of part (i) of Theorem 4.1 is the fact that the complement of $S_o(a)$ is an open set containing zero, and, since $S_o(a)$ is bounded, its complement is non-empty. Thus if $a > 0$, there exists a positive number $\delta = \delta(a)$ such that the open interval $(-\infty,\delta)$ is contained in $S_o(a)^C$. In particular, if $a > 0$ and $b \in (-\infty,\delta)$ then the solution $x(t)$ of (4.1) satisfying $x(0) = a$, $x'(0) = b$ must vanish at least once on $(0,\infty)$. (Recall a similar situation in the case of equation (2.28) above.) If $a = 0$ however, it may happen that for all sufficiently small $b > 0$ the solution of (4.1) satisfying $x(0) = 0$, $x'(0) = b$ will fail to vanish again on $(0,\infty)$, as indicated by the following result which is essentially due to Moore and Nehari [16].

<u>Theorem 4.2</u>

Suppose $\int_o^\infty t^{\gamma+1}q(t)dt < \infty$ where $\gamma > 0$. Then for all

sufficiently small $b > 0$ the solution $x(t)$ of $x'' + q(t)x^{\gamma+1} = 0$ satisfying $x(0) = 0$, $x'(0) = b$ is positive on $(0,\infty)$ and satisfies $\lim_{t \to \infty} x'(t) > 0$.

Proof.

Choose $b > 0$ so small that

$$b^{\gamma} \int_o^{\infty} t^{\gamma+1} q(t)dt < 1 \quad .$$

Suppose $x(t)$ has a first zero on $(0,\infty)$ at the point $t_o > 0$ with $x(t) > 0$ for $t \in (0,t_o)$. Since $x''(t) = -q(t)x^{\gamma+1}(t)$ then $x(t)$ is concave on $(0,t_o)$ and it follows that $0 \leq x(t) \leq bt$ for $t \in [0,t_o)$. Thus

$$b = x'(t) + \int_o^t q(s)x^{\gamma+1}(s)ds \ , \qquad 0 \leq t < t_o$$

and therefore if $t \in [0,t_o)$,

$$0 < b \leq x'(t) + b[b^{\gamma} \int_o^t s^{\gamma+1} q(s)ds] \ ,$$

$$0 < b \leq x'(t) + b[b^{\gamma} \int_o^{\infty} s^{\gamma+1} q(s)ds] < x'(t) + b \quad . \tag{4.8}$$

Therefore $0 < b < x'(t) + b$ for $t \in [0,t_o)$ from which it follows that $x'(t) > 0$ on $[0,t_o)$, contradicting the fact that $x(t_o) = 0$. Therefor $x(t) > 0$ on $(0,\infty)$. To complete the proof we observe that since $x'(t)$ is monotonically decreasing and since, by (4.8),

$$x'(t) \geq b[1 - b^{\gamma} \int_{o}^{\infty} s^{\gamma+1} q(s)ds] > 0$$

then

$$\lim_{t \to \infty} x'(t) \geq b[1 - b^{\gamma} \int_{o}^{\infty} s^{\gamma+1} q(s)ds] > 0 .$$

If $S_o(a) \neq \phi$ for some value of a then according to lemma 3.10 and the comments preceeding Theorem 4.2, the set $S_o(a)$ is bounded above and below by positive numbers depending on a. Moreover, since $S_o(a)$ is closed then either $S_o(a) = \phi$ or the numbers

$$0 < \phi(a) = \inf \{b/b \in S_o(a)\}$$

and

$$\psi(a) = \sup \{b/b \in S_o(a)\}$$

both exist and belong to $S_o(a)$.

In the example of equation (2.28) discussed earlier, it was shown that for each a satisfying $0 < |a| \leq (n+1)^{1/2n}$, the set $S_o(a)$ was non-void and, in fact, $S_o(a)$ consisted of the closed interval $\{b/|b - \frac{a}{2}| \leq \frac{|a|}{2}(1 - \frac{a^{2n}}{n+1})^{1/2}\}$. We have shown above that for the equation (4.1), subject to (4.4) and (4.5), if $S_o(a) \neq \phi$ for some $a > 0$ then

$$S_o(a) \subseteq [\phi(a), \psi(a)] .$$

It remains an open question whether in general set equality holds in the above statement.

It was also shown above that for equation (2.28) the sets $S_o(a)$ are non-empty for all a satisfying $0 < |a| \leq (n+1)^{1/2n}$. We now wish to show that for equation (4.1), if (4.4) and (4.5) are satisfied, then under certain additional conditions on $f(x)$, the set of those initial values $a > 0$ for which $S_o(a)$ is non-empty is either void or consists of a half closed interval of the form $(0,A^+]$ for some $A^+ > 0$.

Suppose (4.4) and (4.5) are satisfied, and let $T > 0$ be chosen. By lemma 3.10 there exists a constant $M > 0$ such that if either $|a| > M$ or $|b| > M$ then the solution $x(t)$ satisfying (4.1) and the initial conditions

$$x(0) = a \ , \quad x'(0) = b \tag{4.9}$$

will have a zero at some point $T_1 \in (0,T)$. Therefore, $S_o(a) = \phi$ for $|a| > M$. Thus either $S_o(a) = \phi$ for every a (i.e., (4.1) fails to admit zero-free solutions) or for some $M > 0$,

$$S_o(a) \neq \phi \quad \text{implies} \quad 0 < |a| < M \ , \tag{4.10}$$

and in this case

$$b \in S_o(a) \quad \text{implies} \quad 0 < |b| < M \ . \tag{4.11}$$

If, therefore, there exists some $a > 0$ for which $S_o(a) \neq \phi$, then the set

$$S_o^+ = \{a > 0/S_o(a) \neq \phi\} \tag{4.12}$$

is bounded above by M and below by zero. Similarly, if there exists some $a < 0$ for which $S_o(a) \neq \phi$, then the set

$$S_o^- = \{a < 0 / S_o(a) \neq \phi\} \tag{4.13}$$

is bounded below by $-M$ and above by zero. Moreover, it follows from (4.11) that if $S_o(a) \neq \phi$ for some $a > 0$ then $\psi(a) < M$, while if $S_o(a) \neq \phi$ for some $a < 0$ then $|\phi(a)| < M$. One concludes, therefore, that if zero-free solutions to equation (4.1) exist, then the set of those initial conditions (a,b) for which the solution of (4.1)-(4.9) is free of zeros, is a subset of $[-M,M] \times [-M,M]$ in R^2 for some constant M. Finally, if $S_o^+ \neq \phi$ (respectively if $S_o^- \neq \phi$) then there exists a number A^+ (respectively A^-) such that $0 \leq \inf S_o^+ \leq \sup S_o^+ = A^+$ (respectively $A^- = \inf S_o^- \leq \sup S_o^- \leq 0)$.

<u>Theorem 4.3</u>

Let (4.4) and (4.5) hold and assume that $S_o^+ \neq \phi$. Then (i) $A^+ \in S_o^+$. If, in addition,

$$\begin{aligned} &f'(u) \text{ is continuous for } |u| < \infty, \text{ and} \\ &\text{monotone for } 0 \leq u < \infty, \end{aligned} \tag{4.14}$$

then (ii) $S_o^+ = (0,A^+]$, (therefore $\inf S_o^+ = 0$).

<u>Proof</u>.

(i) If S_o^+ is a finite set then $A^+ = \sup S_o^+ = \max S_o^+ \in S_o^+$. We may therefore assume that S_o^+ is not finite. If A^+ is an isolated point of S_o^+ then clearly $A^+ \in S_o^+$. Thus we need consider only the case in which A^+ is a limit point of S_o^+. In this case, there exists a sequence $\{\epsilon_n\}$ of positive numbers, strictly decreasing to zero, such that $A^+ - \epsilon_n$ is an element of S_o^+ for each $n = 1,2,3,\ldots$. Then

$S_o(A^+ - \epsilon_n) \neq \phi$ for each n, so there exists $b_n \in S_o(A^+ - \epsilon_n)$. Since

$$0 < \phi(A^+ - \epsilon_n) \leq b_n \leq \psi(A^+ - \epsilon_n) < M ,$$

the sequence $\{b_n\}$ is bounded above and below, and therefore admits a convergent subsequence, which for simplicity we denote by $\{b_j\}$, whose limit $b = \lim_{j \to \infty} b_j$ satisfies $0 \leq b \leq M$. Let $x_j(t)$ and $x(t)$ solve (4.1) and satisfy

$$x_j(0) = A^+ - \epsilon_j , \quad x_j'(0) = b_j ;$$

$$x(0) = A^+ , \quad x'(0) = b , \text{ respectively.}$$

We shall show that $x(t)$ does not vanish on $[0,\infty)$, (which implies $b > 0$), in which case we may conclude that $b \in S_o(A^+)$ and hence that $A^+ \in S_o^+$. The proof that S_o^+ contains any given cluster point is similar.

Suppose $x(t)$ has a zero at some point $t_1 > 0$, with $x(t) > 0$ on $(0,t_1)$. Choose $T > t_1$ such that $x'(t) < 0$ on $(t_1,T]$. Then (by Theorem 3.2, page 14 of Hartman [10]) there exists a subsequence of $\{x_j(t)\}$ which converges uniformly on $[0,T]$ to $x(t)$. But each function in the subsequence is strictly positive on $[0,\infty)$, whereas $x(t)$ is negative at T, which is a contradiction.

For the proof of (ii) we make use of the following theorem of Erbe.

<u>Theorem 4.4</u> (Erbe [9], Theorem 4.6 and Corollary 4.7)

Consider the differential equation

$$x'' = f(t,x) \tag{4.15}$$

where $f(t,x)$ and $f_x(t,x)$ are continuous on $[a,b] \times R$. Assume there exist $C^{(2)}[a,b]$ functions $\alpha(t)$, $\beta(t)$ such that $\alpha'' \geq f(t,\alpha)$, $\beta'' \leq f(t,\beta)$ and $\alpha(t) < \beta(t)$ on $[a,b]$. Assume further that $f_y(t,y)$ is monotone in y for $\min \alpha(t) < y < \max \beta(t)$. Then for any $\alpha(a) < c < \beta(a)$, $\alpha(b) < d < \beta(b)$ there is a solution $x_o(t)$ of (4.15) satisfying $x_o(a) = c$, $x_o(b) = d$ and $\alpha(t) < x_o(t) < \beta(t)$ on $[a,b]$.

For the equation (4.1) $x'' = -q(t)f(x)$ our assumptions on $q(t)$ and (4.14) imply that $-q(t)f'(y)$ is continuous for all y and that $-q(t)f'(y)$ is monotone in y for $0 \leq y < \infty$. Moreover, by Theorem 4.3 (i) the strictly increasing solution $x_*(t)$ of (4.1) satisfying the initial conditions $x_*(0) = A^+$, $x_*'(0) = \phi(A^+)$, and the identically zero solution of (4.1) satisfy the requirements on $\beta(t)$ and $\alpha(t)$ respectively of Theorem 4.4 on any compact interval $[0,T]$. Now let the number a be chosen in $0 < a < A^+$. By Theorem 4.4 there exists a solution $x_n(t)$ of (4.1) satisfying

$$\left.\begin{aligned} & x_n(0) = a\,, \qquad x_n(n) = x_*(n) - \frac{x_*(n) - A^+}{n} \\ & 0 < x_n(t) < x_*(t) \quad ; \quad t \in [0,n] \quad . \end{aligned}\right\} \tag{4.16}$$

Because $0 < x_n(t) < x(t)$ for t in $[0,n]$, then, in fact

$$0 < a \leq x_n(t) < x_*(t) \quad \text{for} \quad t \text{ in } [0,n] \,, \tag{4.17}$$

otherwise, since $x_n''(t) < 0$ for $x_n(t) > 0$, $x_n'(t)$ would have to become negative at some point in $(0,n)$ forcing $x_n(t)$ to vanish at least

twice in that interval, contradicting Theorem 4.4. Moreover, the functions $x_n(t)$ are solutions of (4.1) on the entire interval $[0,\infty)$. Let $I_n = [0,n]$.

On the interval I_1, each solution $x_n(t)$ is positive, increasing in t, bounded above by $x_*(t)$ and satisfies $x_n''(t) \leq 0$. Thus $x_n'(t) \leq x_n'(0)$ on I_1 and, by lemma 3.10, there exists $R_1 = R_1(1)$ such that $x_n'(0) < R_1$ for all n. Therefore the family $F_1 = \{x_n(t)\}$ is uniformly bounded and equicontinuous on I_1:

$$|x_n(t_2) - x_n(t_1)| \leq \int_{t_1}^{t_2} |x_n'(t)|dt \leq R_1(t_2-t_1) ,$$

$$x_n(t) \leq x_n(0) + x_n'(0)t \leq a + R_1 , \qquad t \in [0,1] .$$

Thus, by the Ascoli lemma (Coddington and Levinson [4], p. 5), F_1 contains a subsequence $F_1^{(1)} = \{x_n^{(1)}(t)\}$ which is uniformly convergent on I_1 to a continuous function $x^{(1)}(t)$ satisfying $x^{(1)}(0) = a$ and, by (4.17), $a \leq x^{(1)}(t) \leq x_*(t)$ on I_1. Now $0 < x_n'(0) < R_1$ for all n so also $0 < x_n^{(1)\prime}(0) < R_1$ for all n. Thus $\{x_n^{(1)\prime}(0)\}$ has a subsequence $\{x_{n_k}^{(1)\prime}(0)\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} x_{n_k}^{(1)\prime}(0) = b \tag{4.18}$$

for some $b \in [0,R_1]$. But the subsequence $x_{n_k}^{(1)}(t)$ also converges uniformly to $x^{(1)}(t)$ on I_1. Now on I_1

$$x_n^{(1)}(t) = a + x_n^{(1)\prime}(0)t - \int_o^t \left\{\int_o^s q(u)f(x_n^{(1)}(u))du\right\} ds . \tag{4.19}$$

But f is uniformly continuous on $[a,x_*(1)]$ and therefore $f(x^{(1)}_{n_k}(t)) \to f(x^{(1)}(t))$ uniformly on I_1. Replacing $x^{(1)}_n$ by $x^{(1)}_{n_k}$ in (4.19) and taking the limit, we then have

$$x^{(1)}(t) = a + bt - \int_0^t \left\{ \int_0^s q(u)f(x^{(1)}(u))du \right\} ds \qquad (4.20)$$

and therefore $x^{(1)}(t)$ is a solution of (4.1). But $0 < a \le x^{(1)}(t) \le x_*(t)$ on I_1 and therefore $b > 0$. Moreover, since on I_1

$$x_n^{(1)'}(t) = x_n^{(1)'}(0) - \int_0^t q(s)f(x_n^{(1)}(s))ds \qquad (4.21)$$

then for $t \in I_1$,

$$\lim_{n_k \to \infty} x_{n_k}^{(1)'}(t) = b - \int_0^t q(s)f(x^{(1)}(s))ds = x^{(1)'}(t) . \qquad (4.22)$$

Since $x_{n_k}^{(1)'}(t) \ge 0$ on I_1, then also $x^{(1)'}(t) \ge 0$ on I_1. Thus the subsequence $F^{(1)} = \{x^{(1)}_{n_k}(t)\}_{k=1}^{\infty} \subset \{x_n(t)\}_{n=1}^{\infty}$ converges uniformly on I_1 to a solution $x^{(1)}(t)$ of equation (4.1) which satisfies:

$$\begin{cases} x^{(1)}(0) = a , \quad x^{(1)'}(0) = b > 0 ; \\ a \le x^{(1)}(t) \le x_*(t) , \qquad t \in I_1 ; \\ x^{(1)'}(t) \ge 0 , \qquad t \in I_1 . \end{cases}$$

Consider the family $F^{(1)}$. On the interval $I_2 = [0,2]$, each member of the family $F^{(1)}$ is bounded in absolute value by $B_2 = \max \{x_*(2), \max_{t \in I_2} |x_1(t)|\}$ (in fact $|x_n(t)| < x_*(2)$ for $n \ge 2$

while possibly $x_1(t) > x_*(2)$ in (1,2]), while the derivative of each member of $F^{(1)}$ is bounded in absolute value by the number $C_2 = \max\{R_2, \max_{t \in I_2} |x_1'(t)|\}$ where $R_2 = R_2(2)$ is an upper bound for the set $\{x_n'(0)/n = 2,3,4,\ldots\}$ which exists, by lemma 3.10, as a result of the fact that the solutions $x_n(t)$, $n = 2,3,\ldots$ are free of zeros on I_2. Therefore $F^{(1)}$ is uniformly bounded and equicontinuous on I_2, so, as in the above argument, $F^{(1)}$ contains a subsequence $F^{(2)} = \{x_n^{(2)}(t)\}_{n=1}^{\infty}$ which converges uniformly on I_2 to a solution $x^{(2)}(t)$ of (4.1) which, by (4.18) and since $F^{(2)} \subset F^{(1)}$, must satisfy

$$\begin{cases} x^{(2)}(0) = a\ , \quad x^{(2)'}(0) = b > 0\ ; \\ a \leq x^{(2)}(t) \leq x_*(t)\ , \qquad t \in I_2\ ; \\ x^{(2)'}(t) \geq 0\ , \qquad t \in I_2\ ; \end{cases}$$

and therefore $x^{(2)}(t) \equiv x^{(1)}(t)$ by uniqueness of solutions to initial value problems for (4.1).

Continuing in this manner, for each positive integer k we obtain a subsequence $F^{(k)} = \{x_n^{(k)}(t)\}_{n=1}^{\infty} \subset F^{(k-1)}$ which converges uniformly on I_k to a solution $x^{(k)}(t)$ of (4.1) satisfying

$$\begin{cases} x^{(k)}(0) = a\ , \quad x^{(k)'}(0) = b > 0\ ; \\ a \leq x^{(k)}(t) \leq x_*(t)\ , \qquad t \in I_k\ ; \\ x^{(k)'}(t) \geq 0\ , \qquad t \in I_k\ ; \end{cases}$$

and therefore $x^{(k)}(t) \equiv x^{(k-1)}(t) \equiv x^{(1)}(t)$ by the uniqueness of solutions to initial-value problems. Thus for every positive integer k, $x^{(1)}(t)$

is a solution to (4.1) on I_k and satisfies $x^{(1)}(0) = a$, $x^{(1)'}(0) = b$, $a \leq x^{(1)}(t) \leq x_*(t)$, $t \in I_k$. Letting $k \to \infty$, we conclude that $x^{(1)}(t)$ solves (4.1)-(4.9) and $a \leq x^{(1)}(t) \leq x_*(t)$ on $[0,\infty)$. Therefore $a \in S_o^+$ and, since a was chosen arbitrarily in $(0,A^+)$ it follows that $S_o^+ = (0,A^+]$ and hence $\inf S_o^+ = 0$, which completes the proof.

One immediately observes in the preceeding proof that if $\phi(A^+) < \psi(A^+)$ then the proof of Theorem 4.3 (ii) can as well be carried out by making use, in (4.16), of the solution $y(t)$ of (4.1) satisfying initial conditions $y(0) = A^+$, $y'(0) = \psi(A^+)$ instead of the solution $x_*(t)$ satisfying $x_*(0) = A^+$, $x_*'(0) = \phi(A^+)$. One may also conclude from the same proof that the existence of one solution $x^{(a)}(t)$ of (4.1) satisfying

$$x^{(a)}(0) = a > 0 \ , \quad x^{(a)}(t) > 0 \quad \text{on} \quad [0,\infty) \tag{4.23}$$

implies, for each $\alpha \in (0,a]$, the existence of a solution $x_\alpha^{(a)}(t)$ of (4.1) satisfying

$$x_\alpha^{(a)}(0) = \alpha \quad , \quad x_\alpha^{(a)}(t) > 0 \quad \text{on} \quad [0,\infty) \tag{4.24}$$

such that the family $\{x_\alpha^{(a)}(t)/\alpha \in (0,a]\}$ has the property

$$\begin{gathered} \text{if} \quad \alpha_1,\alpha_2 \in (0,a] \quad \text{with} \quad \alpha_1 < \alpha_2 \quad \text{then} \\ \alpha_1 < x_{\alpha_1}^{(a)}(t) < x_{\alpha_2}^{(a)}(t) < x^{(a)}(t) \quad \text{on} \quad (0,\infty) \ . \end{gathered} \tag{4.25}$$

In the example of equation (2.28) discussed above, any family of solutions $\{x_\alpha^{(a)}(t)\}$ for $0 < a \leq (n+1)^{1/2n}$ which has property (4.25) also satisfies the condition

$$\liminf_{\alpha \to 0+} x_\alpha^{(a)'}(0) = 0 \tag{4.26}$$

that is, a sequence $\{x_{\alpha_n}^{(a)}(t)\}$ of such solutions converges to the trivial solution $x \equiv 0$ as $\alpha_n \to 0$.

§4.2 On non-vanishing solutions of $x'' + q(t)x^{2n+1} = 0$

Consider the equation

$$x'' + q(t)x^{2n+1} = 0 \ , \qquad q > 0 \tag{4.27}$$

where n is a positive integer and $q(t)$ is continuous on $[0,\infty)$ and such that solutions of (4.27) exist on $[0,\infty)$. Then, as in the previous section, for each $a > 0$ either $S_o(a) = \phi$ or there exist numbers $\phi(a) > 0$ and $\psi(a) \geq \phi(a)$ such that $S_o(a) \subseteq [\phi(a),\psi(a)]$. The results of this section provide certain estimates on these numbers in terms of the coefficient function $q(t)$.

Lemma 4.5

Let $x(t)$ be the solution of (4.27) satisfying the initial conditions

$$x(0) = a > 0 \ , \quad x'(0) = b \ . \tag{4.28}$$

If $x'(t) > 0$ on $[0,\tau)$ then

$$b > a^{2n+1} \int_o^\tau q(s)ds \ .$$

Proof.

Since $x'(t) > 0$ on $[0,\tau)$ it follows directly from (4.27) that

$$b = x'(\tau) + \int_0^\tau q(s)x^{2n+1}(s)ds \geq \int_0^\tau q(s)x^{2n+1}(s)ds > a^{2n+1}\int_0^\tau q(s)ds \ .$$

Corollary 4.6

If, for $a > 0$,

$$b < a^{2n+1}\int_0^\infty q(s)ds$$

then $b \notin S_o(a)$. Therefore, if $S_o(a) \neq \phi$ for $a > 0$ then

$$\phi(a) \geq a^{2n+1}\int_0^\infty q(s)ds \quad .$$

Proof.

If $b \leq 0$ then clearly $b \notin S_o(a)$. We may therefore suppose that

$$0 < b < a^{2n+1}\int_0^\infty q(s)ds \quad .$$

Since $\int_0^t q(s)ds$ is a continuous and increasing function of t , there exists a number $\tau < \infty$ such that

$$b = a^{2n+1}\int_0^\tau q(s)ds \quad .$$

Thus, by lemma 4.5, the solution $x(t)$ of (4.27)-(4.28) satisfies $x'(t_o) = 0$ for some $t_o \in (0,\tau]$, and hence $x(t)$ must vanish at least

once on $(0,\infty)$. Thus $b \notin S_o(a)$. It therefore follows that

$$\beta \geq a^{2n+1} \int_o^\infty q(s)ds$$

for any $\beta \in S_o(a)$, and thus

$$\phi(a) \geq a^{2n+1} \int_o^\infty q(s)ds \quad .$$

<u>Theorem 4.7</u>

Suppose $q(t)$ is non-increasing. If, for $a > 0$, there exists $t_o \geq 0$ such that

$$\frac{b}{a^{2n+1}} \leq t_o q(t_o) \tag{4.29}$$

then $b \notin S_o(a)$. In particular, for each $a \in (0,A^+]$,

$$\phi(a) \geq a^{2n+1} \sup_{[0,\infty)} tq(t) \quad .$$

<u>Proof</u>.

If $\int_o^\infty tq(t)dt = \infty$ then all solutions oscillate and hence $b \notin S_o(a)$ for all a, b . Therefore, we may assume without loss of generality that $\int_o^\infty tq(t)dt < \infty$.

Let $a > 0$ and suppose $b \in S_o(a)$. Then the solution $x(t)$ of (4.27)-(4.28) is positive for all t . Using Holder's inequality for integrals we obtain, for $t \geq 0$,

$$\int_o^t x(s)ds \leq \left(\int_o^t ds\right)^{\frac{2n}{2n+1}} \left(\int_o^t x^{2n+1}(s)ds\right)^{\frac{1}{2n+1}} = t^{\frac{2n}{2n+1}} \left(\int_o^t x^{2n+1}(s)ds\right)^{\frac{1}{2n+1}} . \tag{4.30}$$

Therefore

$$\int_o^t x^{2n+1}(s)ds \geq t^{-2n}\left(\int_o^t x(s)ds\right)^{2n+1} , \qquad t > 0 .$$

Since $x'(t) > 0$ for all t , then

$$b = x'(t) + \int_o^t q(s)x^{2n+1}(s)ds > \int_o^t q(s)x^{2n+1}(s)ds$$

and, since $q(t)$ is non-increasing, then

$$b > q(t) \int_o^t x^{2n+1}(s)ds \quad . \tag{4.31}$$

Making use of (4.30) in (4.31) gives the result

$$b > t^{-2n}q(t)\left(\int_o^t x(s)ds\right)^{2n+1} , \qquad t > 0$$

and, since $x(t)$ is strictly increasing,

$$b > t^{-2n}q(t)\left(\int_o^t x(0)ds\right)^{2n+1} = a^{2n+1}tq(t) , \qquad t > 0 .$$

Since the above inequality holds for all positive t , we have a contradiction of (4.29). Therefore, since $\phi(a) \in S_o(a)$ whenever $S_o(a) \neq \phi$, we have

$$\phi(a) \geq a^{2n+1} \sup_{[0,\infty)} tq(t) , \qquad a \in (0,A^+] ,$$

which completes the proof.

We remark that each of the results of this section has a corresponding analogue for the case in which the initial value is negative.

They follow directly from the observation that $x(t)$ solves (4.27) if and only if $-x(t)$ is also a solution.

APPENDIX

In the proof of Theorem 2.1 reference is made to the following lemma:

Lemma

Let $q(t)$ be positive and continuous on $[0,\infty)$, and of bounded variation on compact sets. Let $t_1 > 0$ be chosen, and let $m > 0$ be such that $q(t) \geq m$ on $[0,t_1]$. Finally, let T be the total variation of $q(t)$ on $[0,T]$. Then there exists a sequence of functions $p_k(t) \in C^{(1)}[0,t_1]$ such that

(i) $m \leq p_k(t)$, $\quad t \in [0,t_1]$

(ii) $\int_0^{t_1} |p_k'(t)|dt \leq T$,

(iii) $p_k(t)$ converges to $q(t)$ uniformly on $[0,t_1]$.

Proof.

We extend $q(t)$ to a function $f(t) \in C(-\infty,\infty)$ be defining

$$f(t) = \begin{cases} q(0) \ ; & -\infty < t \leq 0 \\ q(t) \ ; & 0 \leq t \leq t_1 \\ q(t_1) \ ; & t_1 \leq t < \infty \ . \end{cases}$$

Then the total variation of f on $(-\infty,\infty)$ is equal to T, i.e., $f \in BV(-\infty,\infty)$. For each positive integer n define

$$K_n(t) = \begin{cases} A_n \exp(-[(nt)^2-1]^{-1}) \ ; & |t| \leq n^{-1} \\ 0 \ ; & |t| > n^{-1} \end{cases}$$

where A_n is chosen such that

$$\int_{-\infty}^{\infty} K_n(t)dt = \int_{-\frac{1}{n}}^{\frac{1}{n}} K_n(t)dt = 1 .$$

Then $K_n(t)$ is differentiable everywhere on $(-\infty,\infty)$. Define

$$p_n(t) = \int_{-\infty}^{\infty} K_n(t-s)f(s)ds .$$

Then, since $K_n(t-s) \equiv 0$ for $|t-s| > \frac{1}{n}$ and $f \in BV(-\infty,\infty)$ (and thus f is differentiable a.e. in $(-\infty,\infty)$) it follows that $p_n(t)$ is differentiable everywhere, and

$$p_n'(t) = \int_{-\infty}^{\infty} K_n'(t-s)f(s)ds = \int_{-\infty}^{\infty} K_n(t-s)df(s) .$$

Thus

$$\begin{aligned} V_o^{t_1} p_n(t) &= \int_o^{t_1} |p_n'(t)|dt = \int_o^{t_1} \left| \int_{-\infty}^{\infty} K_n(t-s)df(s)\right|dt \leq \\ &\leq \int_o^{t_1} \int_{-\infty}^{\infty} K_n(t-s)|df(s)| \, dt = \int_{-\infty}^{\infty} |df(s)| \int_o^{t_1} K_n(t-s)dt \leq \\ &\leq \int_{-\infty}^{\infty} |df(s)| \int_{-\infty}^{\infty} K_n(t-s)dt = \int_{-\infty}^{\infty} |df(s)| = \int_o^{t_1} |df(s)| \leq T , \end{aligned}$$

which establishes (ii). Moreover $f(t) \geq m$ for all t, hence

$$p_n(t) = \int_{-\infty}^{\infty} K_n(t-s)f(s)ds \geq m \int_{-\infty}^{\infty} K_n(t-s)ds = m ,$$

which proves (i).

Now f is uniformly continuous on $[-1,t_1+1]$. Given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x_1,x_2 \in [-1,t_1+1]$ with $|x_1-x_2| < \delta$ then $|f(x_1)-f(x_2)| < \epsilon$. Choose $N > \frac{1}{\delta}$. Then, for any $t \in [0,t_1]$, if $n \geq N$ we have

$$|p_n(t)-f(t)| = \left| \int_{-\infty}^{\infty} K_n(t-s)f(s)ds - f(t)\right| =$$

$$= \left| \int_{-\infty}^{\infty} K_n(t-s)f(s)ds - \int_{-\infty}^{\infty} K_n(t-s)f(t)ds\right| \leq$$

$$\leq \int_{-\infty}^{\infty} K_n(t-s)|f(s)-f(t)|ds = \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} K_n(t-s)|f(s)-f(t)|ds .$$

But $t - \frac{1}{n} \leq s \leq t + \frac{1}{n}$ implies $|s-t| \leq \frac{1}{n} < \delta$. Therefore

$$|p_n(t)-f(t)| < \epsilon \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} K_n(t-s)ds = \epsilon \quad .$$

Thefore (iii) is proved.

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